THE PRINCIPLES

OF THE

DIFFERENTIAL CALCULUS;

WITH ITS APPLICATION TO

CURVES AND CURVE SURFACES.

DESIGNED FOR

THE USE OF STUDENTS

In the University.

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The present performance is a new edition of a Work which was published a short time ago, and became very soon out of print. The circumstances attending its first appearance were the cause of many imperfections, which, it is presumed, have now been removed, and the reception experienced from the public on the former occasion inspires the Author with some degree of confidence as to what he may be justified in expecting on this.

The Introductory Chapter is a short account of the *Method of Limits*, and comprises some important results deduced by means of that Doctrine.

The first Chapter on the Differential Calculus contains the *Definitions* of the Terms employed in this Part of Mathematical Science, and several Conclusions derived from its first Principles.

The second, third and fourth Chapters contain respectively the Investigations of the Differentials and Differential Coefficients of *Algebraical*, *Exponential* and *Logarithmical*, *Trigonometrical* and *Geometrical*, Functions of one principal or independent Variable, and each of them is accompanied with a great variety of illustrative matter.

The fifth Chapter proceeds to some important applications of the Calculus to the *Developement* of Functions by means of the Theorems of *Maclaurin* and *Taylor*, and concludes with concise accounts of the Methods of *Fluxions*, *Indivisibles*, *Infinitesimals*, *Derived Functions* and the *Residual Analysis*.

In the sixth Chapter is explained and copiously exemplified the doctrine of *Indeterminate* or *Vanishing Fractions*.

The seventh Chapter comprises a very full and circumstantial account of the *Maxima* and *Minima* of Functions of one independent Variable, illustrated by a great variety of examples worked out.

The eighth and ninth Chapters treat, at considerable length, of *Plane Curves* referred to Rectangular and Polar Co-ordinates respectively, and comprise much valuable information concerning them not to be found in any other English treatise on the subject.

In the tenth Chapter have been very fully discussed the Analytical Characters of what are called the *Singular Points* of Plane Curves; and in the eleventh the Principles previously established have been applied to the *Describing* or *Tracing* of Curves. This and the preceding Chapters may be said to constitute the first part of the work.

The twelfth Chapter comprises the Differential Calculus as extended to Functions of *two* or *more* independent Variables; and in the thirteenth it has been applied to the determination of their Maxima and Minima.

The fourteenth Chapter treats of the Application of the Differential Calculus to *Curve Surfaces* and *Curves of Double Curvature.*

The Fifteenth Chapter is a collection of *Miscellaneous Theorems* and *Problems* of great interest and importance, all worked out at length, and evincing in much variety, the Powers of the Calculus as an instrument of Analysis.

Though numerous examples, illustrative of their respective subjects, are interspersed throughout most chapters of the work, the importance of Examples for the exercise of the Student has determined the Author shortly to publish a very complete Collection, in which reference is made to the Chapters and Articles of this Volume for the means of their solution. This Collection would have been appended to the present publication, had not the size to which it has already arrived induced the Author to change his mind.

CAMBRIDGE,
October 24, 1831.

C rv

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ERRATA.

Page 9, line 4 from bottom, *for* log (4) *read* by (4).

—— 12, — 2 ————, *for* $4\pi r . AN = 4\pi AP^2$ *read* $2\pi r . AN = \pi AP^2$.

—— 39, — 2 from top, *for* a^x *read* a^{x+h}.

——134, — 9 ————, *for* $u \dfrac{\frac{d^2 P}{dx^2}}{\frac{d^2 Q}{dx^2}}$ *read* $u = \dfrac{\frac{d^2 P}{dx^2}}{\frac{d^2 Q}{dx^2}}$.

——200, — 2 from bottom, *for* $x = 0$ *read* $x' = 0$, and *for* axis of x *read* axis of y.

——224, — 1 from top, *for* Orders of Contact *read* Contact and Osculation.

INTRODUCTORY CHAPTER.

ON THE METHOD OF LIMITS.

Article I. Definition I.

The *Limits* of a quantity which admits of change in its magnitude, are those magnitudes between which all the values that it can have during all its changes are comprised; beyond which it can never pass; and from which it may be made to differ by quantities less than any that can be assigned in finite terms.

Ex. 1. The quantity ax, wherein x admits of all possible values from zero or 0, to infinity or ∞, becomes 0 in the former case, and ∞ in the latter; and consequently the limits of the algebraical expression ax are 0 and ∞: the former being called the *inferior*, and the latter the *superior* limit.

Ex. 2. In the quantity expressed by $ax + b$, if we make $x = 0$, we have the inferior limit equal to the finite quantity b, and if x be supposed $= \infty$, the superior limit is found to be indefinitely great.

Ex. 3. Again, if we take the fraction $\frac{ax+b}{bx+a}$ which is equivalent to the ratio $ax+b : bx+a$, and suppose x to vanish, we find the inferior limit to be $\frac{b}{a}$; but if x be indefinitely increased, the superior limit is $\frac{a+\frac{b}{x}}{b+\frac{a}{x}}$ wherein x is infinite; that is, the superior limit is $\frac{a}{b}$.

Hence in this example, the superior and inferior limits are the reciprocals of each other.

Ex. 4. The sum of the geometrical series $a+\frac{a}{x}+\frac{a}{x^2}+$ &c. continued to n terms, is expressed by the quantity

$$\frac{a\left(\frac{1}{x^n}-1\right)}{\frac{1}{x}-1}=\frac{ax\left(1-\frac{1}{x^n}\right)}{x-1};$$

now, if $n=0$, the inferior limit is manifestly $=0$; but if $n=\infty$, $\frac{1}{x^n}$ becomes 0, and therefore the superior limit is $\frac{ax}{x-1}$; which is usually called the sum of the series continued *in infinitum.*

Ex. 5. The ratio $ax^2+bx : cx^2+dx$ is equal to the ratio $ax+b : cx+d$, and therefore also the limits of the general ratio $ax^2+bx : cx^2+dx$ are the particular ratios $b : d$ and $a : c$.

This example shews that, though the terms of a ratio be evanescent or infinite, its limits may be finite.

Ex. 6. If a regular polygon be inscribed in a circle, and the number of its sides be continually doubled, it is evident

that its perimeter approaches more and more nearly to equality with the periphery of the circle, and that at length their difference must become less than any quantity that can be assigned; hence therefore, the circumference of the circle is the limit of the perimeters of the polygons.

2. *To prove that the Limits of the ratios subsisting between the sine and tangent of a circular arc, and the arc itself, are ratios of equality.*

Let p and p' represent the perimeters of two regular polygons of n sides, the former inscribed in, the latter circumscribed about, a circle whose radius is 1, and circumference = 6.28318 &c. = 2π:

$$\text{then (Trig.) } p = 2n \sin\frac{\pi}{n}, \text{ and } p' = 2n \tan\frac{\pi}{n};$$

$$\text{hence } \frac{p}{p'} = \frac{2n \sin\frac{\pi}{n}}{2n \tan\frac{\pi}{n}} = \cos\frac{\pi}{n},$$

and if the value of n be supposed to be indefinitely increased, the value of $\cos\frac{\pi}{n}$ is 1, and therefore $p = p'$: now, the periphery of the circle evidently lies between p and p', and therefore in this case is equal to either of them; hence on this supposition, an n^{th} part of the perimeter of the polygon is equal to an n^{th} part of the periphery of the circle: that is,

$$2\sin\frac{\pi}{n} = \frac{2\pi}{n} = 2\tan\frac{\pi}{n}, \text{ or } \sin\frac{\pi}{n} = \frac{\pi}{n} = \tan\frac{\pi}{n},$$

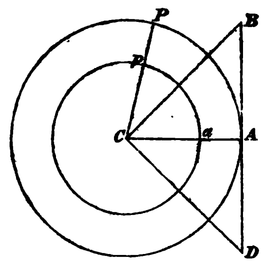
or the sine and tangent of a circular arc in their *ultimate* or *limiting state*, are in a ratio of equality with the arc itself.

3. Cor. Hence also, we shall obviously have

$$\text{chd}\left(\frac{\pi}{n}\right) = 2\sin\left(\frac{\pi}{2n}\right) = 2\left(\frac{\pi}{2n}\right) = \left(\frac{\pi}{n}\right) \textit{ ultimately.}$$

4. *To find the Circumference of a Circle.*

Let BD be the side of an equilateral and equiangular



polygon of n sides, circumscribed about the circle whose radius $CA = r$; then the angle $BCD = \frac{2\pi}{n}$, and we have

$$BD = 2CA \tan ACB = 2r \tan\left(\frac{2\pi}{2n}\right) = 2r \tan\frac{\pi}{n};$$

$$\therefore \text{ the perimeter of the polygon} = 2nr \tan\frac{\pi}{n}:$$

observing now, that as n continually increases, the perimeter of the polygon approaches to the circumference of the circle, if we suppose n to become indefinitely great, then we have seen that in this case $\tan\frac{\pi}{n}$ becomes equal to $\frac{\pi}{n}$; and therefore the periphery or circumference of the circle $= 2nr\frac{\pi}{n} = 2\pi r$.

5. Cor. 1. From the last article, if C and c be the circumferences of two circles whose radii are R and r, and diameters D and d, we have $C = 2\pi R$, $c = 2\pi r$, and thence

$$\frac{C}{c} = \frac{2\pi R}{2\pi r} = \frac{R}{r} = \frac{D}{d};$$

that is, the circumferences of circles are to one another as their radii or diameters.

6. Cor. 2. If the arc AP be called a and the angle $ACP = A$, we shall by *Euclid* VI. 33. have $\frac{2\pi r}{a} = \frac{2\pi}{A}$;

$$\text{and therefore } a = \frac{2\pi r A}{2\pi} = rA,$$

or the arc is equal to the radius multiplied by its corresponding angle.

7. *To find the Area of a Circle.*

Using the figure and notation of (4), we have

$$\text{the area of the triangle } BCD = \frac{CA \,.\, BD}{2} = r^2 \tan\frac{\pi}{n};$$

$$\therefore \text{ the area of the polygon} = n \,\triangle\, BCD = nr^2 \tan\frac{\pi}{n},$$

and taking the limits of both sides of the equation as before, we obtain

$$\text{the area of the circle} = nr^2\,\frac{\pi}{n} = \pi r^2, \text{ or} = \tfrac{1}{2} \times 2\pi r \times r,$$

which is half the rectangle of its circumference and radius.

8. Cor. 1. If A and a be the areas of two circles whose radii are R and r, we have $A = \pi R^2$ and $a = \pi r^2$;

$$\therefore \frac{A}{a} = \frac{\pi R^2}{\pi r^2} = \frac{R^2}{r^2} = \frac{D^2}{d^2} = \frac{C^2}{c^2} \text{ from (5)};$$

or the areas of circles are to one another as the squares of their radii, diameters or circumferences.

9. Cor. 2. If the notation adopted in (6) be retained, we have by *Euclid* vi. 33,

$$\frac{\text{the sector } ACP}{\pi r^2} = \frac{A}{2\pi}, \therefore \text{ the sector } ACP = \frac{Ar^2}{2}$$

$$= \frac{rAr}{2} = \frac{ra}{2}, \text{ as appears from (6)};$$

that is, the area of a circular sector is equal to half the rectangle of its arc and radius.

10. Cor. 3. Hence, if A and a be the arcs AP, ap of two concentric circles cut off by the same radii produced, and h be the distance between them, we have

$$\text{the sector } ACP = \frac{RA}{2}, \quad \text{and the sector } aCp = \frac{ra}{2},$$

$$\therefore \text{ the area } APpa = \frac{1}{2}\{RA - ra\};$$

$$\text{now } R : r :: A : a, \text{ by similar sectors,}$$

$$\therefore R : R - r :: A : A - a,$$

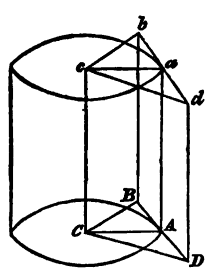
$$\text{and } R = (R-r)\frac{A}{A-a}, \quad r = (R-r)\frac{a}{A-a};$$

$$\text{hence the area } APpa = \frac{1}{2}(R-r)\left\{\frac{A^2}{A-a} - \frac{a^2}{A-a}\right\}$$

$$= \frac{h}{2}\left\{\frac{A^2 - a^2}{A-a}\right\} = \frac{h}{2}(A+a).$$

11. *To find the Surface of a right Cylinder.*

Let $BDdb$ be the side of an equilateral and equiangular prism of n sides circumscribed about the cylinder whose height



is h, and the radius of whose base is r; then the angle $BCD = \frac{2\pi}{n}$; therefore the area of the rectangle $BDdb = BD . Aa$

$$= 2AB . Aa = 2rh \tan\frac{\pi}{n};$$

hence the surface of the prism, neglecting the ends,

$$= n \text{ rectangle } BDdb = 2nrh \tan\frac{\pi}{n}:$$

and taking the limits of both sides of this equation, we shall evidently have the convex surface of the cylinder

$$= 2nrh\frac{\pi}{n} = 2\pi rh,$$

or = to the rectangle of the circumference of the base and the altitude.

12. Cor. Hence the whole surface of a right cylinder = the convex surface + the areas of the two ends

$$= 2\pi rh + 2\pi r^2 = 2\pi r(r+h),$$

which is also the convex surface of a cylinder of the same radius whose altitude is $r+h$.

13. *To find the Content of a right Cylinder.*

Retaining the notation and figure of (11), we have

the content of the triangular prism $BCDdcb$

$$= \text{the area of } \triangle BCD \,.\, Aa = CA \,.\, AB \,.\, Aa = r^2 h \tan\frac{\pi}{n};$$

$\therefore$ the content of the whole circumscribed prism $= nr^2h\tan\frac{\pi}{n}$;

and if the limits be taken, the content of the cylinder

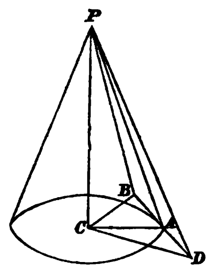
$$= nr^2h\frac{\pi}{n} = \pi r^2 h,$$

or = the area of the base multiplied by the perpendicular altitude.

14. Cor. Hence also the content of a cylinder $= \pi r^2 h$ $= 2\pi r h \times \frac{r}{2}$ = the convex surface multiplied by half the radius.

15. *To find the Surface of a right Cone.*

Let PBD be the side of an equilateral and equiangular pyramid of n sides circumscribed about the cone whose perpen-



dicular altitude is h, side l and the radius of whose base is r; then the angle $BCD = \frac{2\pi}{n}$, and we have

$$\text{the area of the triangle } PBD = AB \, . \, AP = rl \tan \frac{\pi}{n};$$

$\therefore$ the surface of the pyramid, neglecting the base, $= nrl \tan \frac{\pi}{n}$; and the limits being taken, we obtain the convex surface of the cone

$$= nrl \frac{\pi}{n} = \pi r l, \text{ or } = 2\pi r \times \frac{l}{2},$$

which is the circumference of the base multiplied by half the length of the side.

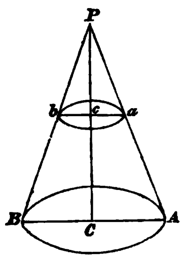
16. Cor. 1. The surface of the cone may be expressed in other terms.

For, $l = \sqrt{r^2 + h^2}$ and therefore the convex surface $= \pi r \sqrt{r^2 + h^2}$; and the whole surface, including the base $= \pi r l + \pi r^2 = \pi r \{r + l\} = \pi r \{r + \sqrt{r^2 + h^2}\}$.

17. Cor. 2. Hence may be found the convex surface of the frustum of a cone.

For, if $AP = L$, $AC = R$, $aP = l$, $ac = r$,

the surface of the whole cone $PCAB = \pi R . PA$;



and the surface of the part $Pcab = \pi r . Pa$;

$\therefore$ the surface of the frustum $= \pi \{R . PA - r . Pa\}$:

but $PA : Pa :: CA : ca :: R : r$, by similar triangles,

$\therefore PA : Aa :: R : R - r$;

$$\text{whence } PA = \frac{R(L-l)}{R-r}, \text{ and } Pa = \frac{r(L-l)}{R-r};$$

$$\therefore \text{ the surface of the frustum } = \pi \left\{ \frac{R^2(L-l)}{R-r} - \frac{r^2(L-l)}{R-r} \right\}$$

$$= \pi(L-l)\left(\frac{R^2-r^2}{R-r}\right) = \pi(L-l)(R+r) = \pi k(R+r),$$

if k be the distance $L - l$ between the circumferences of the ends:

$$\text{or } = \frac{k}{2}\{2\pi R + 2\pi r\} = \frac{k}{2}(C+c) \quad (4)$$

if C and c be those circumferences.

18. Cor. 3. If R be supposed to become $= r$, the limit of the surface of the frustum is $2\pi r k$, which is equal to

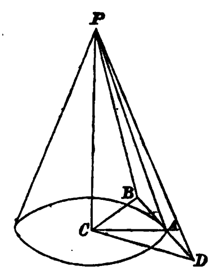
the surface of a cylinder of the same base, and whose altitude is the length of the side.

19. Cor. 4. The whole surface of a conical frustum, including the ends or bases, is equal to

$$\pi k(R+r)+\pi R^2+\pi r^2=\pi\{R(R+k)+r(r+k)\}.$$

20. *To find the Content of a right Cone.*

Since (*Euclid* XII. 7.) every triangular prism may be divided into three equal triangular pyramids, it follows that

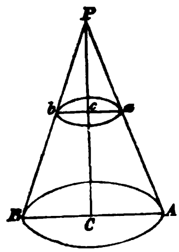


every triangular pyramid is equal to one-third of a triangular prism having the same base and altitude; therefore, we have the content of the triangular pyramid, or

$$PBCD=\tfrac{1}{3}CA.AB.PC=\tfrac{1}{3}r^2h\tan\frac{\pi}{n}:$$

and hence the content of the whole circumscribed pyramid $=\frac{1}{3}nr^2h\tan\frac{\pi}{n}$, from which, as before, we get the content of the cone $=\frac{\pi r^2h}{3}$, or equal to one-third of the cylinder of the same base and altitude.

21. Cor. 1. Hence to find the content of a conical frustum, the radii of whose ends are R and r and height k,



we have the content of the whole cone $PAB = \frac{1}{3}\pi R^2 . PC$

and the content of the part $Pab = \frac{1}{3}\pi r^2 . Pc$;

$$\therefore \text{ the content of the frustum } = \frac{\pi}{3}\{R^2 . PC - r^2 . Pc\}:$$

now $PC : Pc :: CA : ca :: R : r$, by similar triangles;

$$\therefore PC : Cc :: R : R - r;$$

$$\text{hence } PC = \frac{Rk}{R-r}, \text{ and } Pc = \frac{rk}{R-r};$$

$$\therefore \text{ the content of the frustum } = \frac{\pi k}{3}\left\{\frac{R^3 - r^3}{R-r}\right\}$$

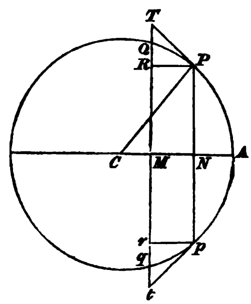
$$= \frac{\pi k}{3}\{R^2 + Rr + r^2\},$$

or = the sum of the areas of the ends and a mean proportional between them, multiplied by one-third of the altitude.

22. Cor. 2. If R be supposed $= r$, the content of the frustum becomes $= \frac{\pi k}{3}(r^2 + r^2 + r^2) = \pi r^2 k$, which is the content of a cylinder, the radius of whose base is r and altitude k: that is, the limit of the frustum of a cone is a cylinder of the same base and altitude.

23. *To find the Surface of a Sphere.*

Let the diameter of the sphere whose radius is r, be divided into n equal parts, of which let NM be one, and



therefore $= \dfrac{2r}{n}$: and about the zone included between two parallel sections passing through N and M, let a conical frustum be described, and put $\angle PCN = \theta$: then by (17) the surface of this frustum

$$= \pi(PN + TM)\, PT = \pi(PN + TM)\, NM \operatorname{cosec} \theta;$$

whence taking the limits, observing that TM is ultimately equal to PN or $r \sin \theta$, we have the surface of the zone corresponding to $NM = \dfrac{4\pi r^2}{n} \sin \theta \operatorname{cosec} \theta = \dfrac{4\pi r^2}{n}$, which not involving θ, is the same for each of the equal parts of the diameter;

$$\therefore \text{ the whole surface of the sphere} = n \frac{4\pi r^2}{n} = 4\pi r^2.$$

24. Cor. 1. If the radius of the sphere be represented by unity, its surface will be represented by 4π, or by twice the circumference of the generating circle.

25. Cor. 2. The surface of the spherical segment $APNp = 4\pi r . AN = 4\pi AP^2 =$ the area of a circle whose radius is equal to the chord of half the corresponding arc.

26. Cor. 3. Hence it is manifest, that the surface of a sphere is equal to the convex surface of its circumscribed cylinder; and also, that the surface of any spherical zone is equal to the surface of the circumscribed cylinder included between the same planes produced, and perpendicular to its axis.

27. *To find the Content of a Sphere.*

Suppose a polyhedron of n faces to be circumscribed about the sphere, and let the area of each of its surfaces be called A: then the sum of the contents of the corresponding solids formed by drawing lines from the centre to the angular points, or the content of the polyhedron $= \frac{nAr}{3}$ by (20): and taking the limits, as before, we have the content of the sphere $= \frac{1}{3}$ surface $\times$ radius $= \frac{1}{3} 4\pi r^2 r = \frac{4\pi r^3}{3}$.

28. Cor. 1. Hence, if S be the surface of any spherical segment whose radius is r, the content of the corresponding spherical sector will be $\frac{1}{3} Sr = \frac{rS}{3}$.

29. Cor. 2. Since the content of the circumscribed cylinder $= \pi r^2 h = \pi r^2 2r = 2\pi r^3$, we have

the sphere : its circumscribed cylinder :: 2 : 3.

DIFFERENTIAL CALCULUS.

CHAP. I.

Definitions and Preliminary Observations.

1. Def. 1. In this science the quantities employed are distinguished into *constant* and *variable*, in the same manner as those in Algebra were into known or given, and unknown or required.

2. Def. 2. Constant or invariable quantities are such as do not change their values during the whole course of each of the operations in which they are employed, and are usually denoted by the former letters of the alphabet A, B, C, &c. L, M, N; a, b, c, &c. l, m, n.

Ex. In the equation to the circle which is $y = \sqrt{2ax - x^2}$, the quantity a, which represents the radius of the circle, undergoes no change in any one operation in which this equation is concerned.

3. Def. 3. Variable quantities are such as admit of different degrees of magnitude in the same expression, according to the manner in which they are involved in it, and are generally expressed by the latter letters of the alphabet P, Q, R, &c. X, Y, Z; p, q, r, &c. x, y, z.

Ex. In the equation just mentioned, the quantities x and y admit of different magnitudes within certain limits; thus, if

$$x = \frac{a}{2}, y = \sqrt{a^2 - \frac{a^2}{4}} = \frac{\pm a\sqrt{3}}{2}; \text{ if } x = a, y = \sqrt{2a^2 - a^2} = \pm a,$$

$$\text{and if } x = 2a, y = \sqrt{4a^2 - 4a^2} = 0, \text{ \&c.}$$

4. DEF. 4. An analytical expression of any form whatever involving constant and variable quantities, is called a *Function* of those *variable* quantities, without any regard to the constant quantities, or to the manner in which they are involved.

Ex. 1. The quantities $ax^2 + bx + c$, $ax^n - bx^{n-1} + cx^{n-2} - \text{\&c.}$, $\sqrt{2ax - x^2}$, a^x, $\log(ax + b)$, $\sin(2a - 3x)$, &c. are all functions of x: and when a function of x is spoken of *generally*, without regard to its particular nature, it is expressed in one of the following forms, $f(x)$, $F(x)$, $\phi(x)$, $\psi(x)$, &c.

Ex. 2. The algebraical expressions $x^2y^2 + a^2xy + y^4$, $y^n - (ax + b)\,y^{n-1} + (cx^2 + ex + f)\,y^{n-2} - \text{\&c.}$, a^{x+y}, $\log(ax - by)$, $\sin(mx + ny)$, &c. are all functions of x and y, and are expressed in general language by

$$f(x, y),\ F(x, y),\ \phi(x, y),\ \psi(x, y), \text{ \&c.}$$

Similar observations may be made, whatever be the number of variable quantities involved in an expression.

5. DEF. 5. If the relation between x and y be expressed by an equation of the form $y = f(x)$, or $y = F(x)$, &c., y is styled an *explicit* function of x, x being called the *principal* or *independent* variable, and y the *dependent* variable.

Ex. 1. In the equation $y = \sqrt{2ax - x^2}$, y is an explicit function of x: and since the value of y depends upon that of x by means of this equation, x and y are called the independent and dependent variables respectively: also, by the solution of a quadratic, we have $x = a \pm \sqrt{a^2 - y^2}$, in which x is an

explicit function of y, and in this case y is the independent and x the dependent variable.

From this it appears, that if an equation express the relation between two variable quantities, either of them may be considered as a function of the other.

Ex. 2. Similarly, $u = ax^2 + bx + c$, $u = a^x$, $u = \log(x + a)$, $u = \sin(x - a)$, &c. are examples in which u is an explicit function of x.

6. Def. 6. If the relation between two variable quantities x and y be expressed by an equation of the form $f(x, y) = 0$, then may either of the quantities be considered an *implicit* function of the other, one of them being the *independent*, and the other the *dependent* variable.

Ex. 1. The equation $x^2 + y^2 - a^2 = 0$ is a simple instance in which either x or y may be considered an *implicit* function of the other; but by solving the equation with respect to y, we have $y = \pm\sqrt{a^2 - x^2}$, and with respect to x, we get $x = \pm\sqrt{a^2 - y^2}$, in which y and x are each exhibited as an *explicit* function of the other.

From this it is manifest, that an implicit function may be made an explicit one, whenever the equation can be solved with respect to one or both of the variables involved.

Ex. 2. The expressions $u^2 + mux + a^2 + bx + ux^2 = 0$, $u^3 + x^3 - a^2x + a^2u = 0$, $u^2 + x^2 - a^{\frac{u}{x}} = 0$, &c. are instances in which either u or x is an implicit function of the other: by the solutions of a quadratic and cubic equation in the first two cases, one of the variables may be made an explicit function of the other; in the last the solution cannot be effected.

7. Def. 7. To express the functions of any quantity however formed as of x^2, $(a + bx)$, &c. the notation mentioned in (4) is usually extended; thus if y be a quantity whose

value depends upon either of the above mentioned expressions, it is written $y = f(x^2)$, $y = f(a + bx)$, &c.

Ex. As instances of this definition, $y = (a^2 + x^2)(b^2 - x^2)$ is at the same time a function of x and x^2; $y = (a + bx)^m \log(a + bx)$ is an explicit function of $(a + bx)$; &c.

8. Def. 8. If the value of one quantity as u depend upon the magnitudes of two or more others as x, y, z, &c., that quantity is called a function of two, three or more variables, and is written $u = f(x, y, z, \&c.)$, or $u = F(x, y, z, \&c.)$, &c., u being, as before, distinguished by the name of *dependent* variable, and x, y, z, &c. by the titles of *principal* or *independent* variables.

9. Def. 9. All quantities or functions concerned in this subject, are distinguished into three kinds; *Algebraical*, *Transcendental* and *Intranscendental*.

10. Def. 10. Algebraical functions are such as are expressed in rational forms, or can be reduced to such forms in finite numbers of terms.

Ex. 1. The quantities

$$ax + b,\ ax^4 + bx^2 - \frac{c}{x^2},\ \frac{b^2}{a^2}(2ax - x^2),\ ax^m - bx^{m-1} + cx^{m-2} - \&c.,$$

are all algebraical functions according to the former part of the definition.

Ex. 2. The expressions

$$\sqrt{a^2 - x^2},\ \sqrt[3]{2ax + x^2},\ \sqrt[m]{ax^{2n} - bx^n + c},\ \&c.$$

are also algebraical functions, because reducible by the operations of Algebra to rational forms in finite numbers of terms.

Hence it appears that algebraical functions may be either rational or irrational, or partly both.

11. Def. 11. Transcendental functions are such as cannot be exhibited by means of algebraical expressions of finite

numbers of terms, and are either *Exponential, Logarithmical* or *Trigonometrical.*

Ex. Thus a^x, log $(ax+b)$ and sin $(a+x)$ are all transcendental functions, the first being exponential, the second logarithmical and the third trigonometrical.

12. Def. 12. Intranscendental functions are such as have surds &c. for their indices: these functions are of very rare occurrence, and need no exemplification.

13. Def. 13. The *Increment* or *Difference* of any function is the difference between two values of the function, corresponding to certain values of the independent variable, and is generally denoted by the Greek letter Δ placed before the function.

Ex. If $u = ax^2 + bx + c$, and u' be the value of the same function when x becomes x', we have

$u' - u = (ax'^2 + bx' + c) - (ax^2 + bx + c) = a(x'^2 - x^2) + b(x' - x)$:

that is, if the difference of the function or $u' - u$ be denoted by Δu, we have

$$\Delta u = a\,(x'^2 - x^2) + b\,(x' - x);$$

similarly, $x' - x$ is abbreviatedly written Δx, and the ratio of the increments of u and x, is expressed as follows:

$$\frac{\Delta u}{\Delta x} = a\left(\frac{x'^2 - x^2}{x' - x}\right) + b\left(\frac{x' - x}{x' - x}\right) = a\,(x' + x) + b.$$

14. Def. 14. The *Differential Calculus* is that part of Mathematics which treats of the method of finding the limits of the ratios of the simultaneous increments of functions, and of the variables on which they depend.

Ex. In the example $u = ax^2 + bx + c$ last given, we have seen that the ratio of the simultaneous increments of the function and independent variable or $\dfrac{\Delta u}{\Delta x}$ is $a\,(x' + x) + b$, and

the object of the differential calculus is to find the limit of this quantity, or its value when the dependent magnitudes Δx and Δu are indefinitely diminished.

15. DEF. 15. Any quantities whatever taken in the ultimate or limiting ratio of the simultaneous increments of a function and the variable upon which it depends are called the corresponding *Differentials* of the function and independent variable.

EX. If h and k represent the increments of x and u respectively, and if the ultimate value of the fraction $\frac{\Delta u}{\Delta x}$ or of $\frac{k}{h}$ be $= \frac{k_1}{h_1} = \frac{mk_1}{mh_1}$; then may k_1 and h_1, or mk_1 and mh_1 be called the corresponding differentials of u and x; so that, though the notion of a differential be essentially relative, we may still regard it as an absolute but indeterminate magnitude.

16. DEF. 16. Instead of taking different letters to denote the terms of the ultimate ratio above alluded to, the same letters u and x are retained with the small letter d placed before them, to denote what are expressed in the last example by the quantities mk_1 and mh_1; and the particular value of $\frac{\Delta u}{\Delta x}$, which it is the business of this Calculus to ascertain, is written $\frac{du}{dx}$, and du and dx are styled the differentials of u and x respectively.

17. DEF. 17. The quantity represented by $\frac{du}{dx}$ which expresses the limit of the ratio of the simultaneous increments of u and x is called the *Differential Coefficient* of u, because it is the multiplier which connects the differential of the function u with that of the independent variable x.

18. The last five definitions will be better understood, after a perusal of the following examples, whose differentials are derived from first principles.

Ex. 1. In the function $u = ax$, let u be changed into u' in consequence of its principal variable x being changed into x', then we have

$$u' = ax' \text{ and } u = ax;$$

$$\therefore\ u' - u = ax' - ax = a(x' - x), \text{ that is, } \Delta u = a\Delta x,$$

and the equation $\frac{\Delta u}{\Delta x} = a$, expresses the relation between the simultaneous increments of u and x; therefore taking the limits of both sides of this equation, observing that the particular value of $\frac{\Delta u}{\Delta x}$ sought after, is written $\frac{du}{dx}$, and that a does not undergo any change, we shall have

$$\frac{du}{dx} = a, \text{ and } \therefore\ du = adx.$$

Ex. 2. Let $u = ax^3 - bx^2 + cx - e$, then using the notation already pointed out, we have

$$u' = ax'^3 - bx'^2 + cx' - e \text{ and } u = ax^3 - bx^2 + cx - e,$$

$$\therefore\ u' - u = a(x'^3 - x^3) - b(x'^2 - x^2) + c(x' - x)$$

$$= a(x' - x)(x'^2 + x'x + x^2) - b(x' - x)(x' + x) + c(x' - x),$$

$$\text{and } \frac{u' - u}{x' - x} \text{ or } \frac{\Delta u}{\Delta x} = a(x'^2 + x'x + x^2) - b(x' + x) + c,$$

which expresses generally the ratio of the increments of u and x.

Now, to find the limit of this ratio, we observe that if the increment of x be diminished *sine limite*, x' becomes $= x$, and therefore $\frac{du}{dx} = 3ax^2 - 2bx + c$, which expresses the relation of the differentials of u and x: and hence in this case,

$$du = 3ax^2dx - 2bxdx + cdx.$$

Ex. 3. If we have $u = \frac{a^2 + x^2}{a - x}$, then will $u' = \frac{a^2 + x'^2}{a - x'}$;
and therefore

$$u' - u = \frac{a^2 + x'^2}{a - x'} - \frac{a^2 + x^2}{a - x} = \frac{a(x'^2 - x^2) - xx'(x' - x) + a^2(x' - x)}{(a - x')(a - x)}:$$

$$\text{whence } \frac{u' - u}{x' - x} \text{ or } \frac{\Delta u}{\Delta x} = \frac{a(x' + x) - xx' + a^2}{(a - x')(a - x)},$$

which is the ratio of the increments of u and x: wherefore taking the inferior limits of both sides of this equation agreeably to the object of the Calculus and observing that x' is ultimately equal to x, we obtain $\frac{du}{dx} = \frac{a^2 + 2ax - x^2}{(a - x)^2}$, and thence

$$du = \frac{(a^2 + 2ax - x^2)}{(a - x)^2} dx = \frac{a^2 dx + 2axdx - x^2 dx}{(a - x)^2},$$

which is the differential of the function proposed.

Ex. 4. Let $u = \sqrt{a^2 + x^2}$, then as before $u' = \sqrt{a^2 + x'^2}$,

$$\text{and } \frac{u' - u}{x' - x} \text{ or } \frac{\Delta u}{\Delta x} = \frac{\sqrt{a^2 + x'^2} - \sqrt{a^2 + x^2}}{x' - x}$$

$$= \frac{(a^2 + x'^2) - (a^2 + x^2)}{(x' - x)(\sqrt{a^2 + x'^2} + \sqrt{a^2 + x^2})} = \frac{x' + x}{\sqrt{a^2 + x'^2} + \sqrt{a^2 + x^2}},$$

whereof the limits being taken by making $x' = x$, we get

$$\frac{du}{dx} = \frac{2x}{2\sqrt{a^2 + x^2}} = \frac{x}{\sqrt{a^2 + x^2}},$$

which is the general ratio of the differentials of u and x; and if $x = a$, this becomes equal to the particular ratio $\frac{1}{\sqrt{2}}$:

Ex. 5. If we have $u = ax^m$, then will $u' = ax'^m$, and $u' - u = a(x'^m - x^m) = a(x' - x)\{x'^{m-1} + x'^{m-2}x + x'^{m-3}x^2 + \&c.$ to m terms$\}$;

$$\therefore \frac{u'-u}{x'-x} \text{ or } \frac{\Delta u}{\Delta x} = a\{x'^{m-1} + x'^{m-2}x + x'^{m-3}x^2 + \&c. \text{ to } m \text{ terms}\};$$

and taking the limits of both sides of this equation, observing that x' ultimately becomes x, we have

$$\frac{du}{dx} = a\{x^{m-1} + x^{m-1} + x^{m-1} + \&c. \text{ to } m \text{ terms}\}$$

$$= max^{m-1}, \text{ and } \therefore du = max^{m-1}dx:$$

and a similar process will lead to the ratio of the differentials of u and x when the index is negative or fractional.

Ex. 6. Let us take the exponential function $u = a^x$, where a may be assumed of any magnitude whatever, then $u' = a^{x'}$,

$$\therefore u' - u = a^{x'} - a^x = a^x\{a^{x'-x} - 1\}:$$

but $a^{x'-x} = \{1 + (a-1)\}^{x'-x} = 1 + (x'-x)(a-1)$ $+ \dfrac{(x'-x)(x'-x-1)}{1.2}(a-1)^2 + \&c.$ by the binomial theorem:

$$\therefore \frac{u'-u}{x'-x} \text{ or } \frac{\Delta u}{\Delta x} = a^x\left\{(a-1) + \frac{(x'-x-1)}{1.2}(a-1)^2 + \&c.\right\};$$

whence, taking the limits as before, we get

$$\frac{du}{dx} = a^x\{(a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \&c.\} = ka^x,$$

if k denote the quantity between the brackets; and therefore $du = ka^x dx$, which is the differential of u expressed in terms of the differential of x.

Ex. 7. From the trigonometrical function $u = a \sin 2x$, we obtain $u' = a \sin 2x'$,

and therefore,

$$u' - u = a \sin 2x' - a \sin 2x = 2a \cos (x' + x) \sin (x' - x),$$

$$\text{whence } \frac{u' - u}{x' - x} \text{ or } \frac{\Delta u}{\Delta x} = 2a \cos (x' + x) \frac{\sin (x' - x)}{x' - x},$$

which expresses the ratio of the simultaneous increments of u and x: and since, by (2) of the Introductory Chapter, an arc and its sine are equal in the limit, we have

$$\frac{du}{dx} = 2a \cos 2x, \text{ and } \therefore du = 2a \cos 2x dx.$$

Ex. 8. If there be proposed the equation $u^2 - 2ux + a^2 = 0$, wherein u is an implicit function of x, then as before

$$u'^2 - 2u'x' + a^2 = 0:$$

$$\text{whence } u'^2 - u^2 - 2(u'x' - ux) = 0,$$

$$\text{or } (u' + u)(u' - u) - 2\{u'(x' - x) + x(u' - u)\} = 0;$$

$$\therefore (u' + u)\frac{u' - u}{x' - x} - 2\left\{u' + x\frac{u' - u}{x' - x}\right\} = 0,$$

that is, the equation

$$(u' + u)\frac{\Delta u}{\Delta x} - 2\left\{u' + x\frac{\Delta u}{\Delta x}\right\} = 0,$$

expresses the general relation of the differences or increments of u and x; therefore if we take the limits, observing that u' is ultimately equal to u, we shall have

$$2u\frac{du}{dx} - 2\left\{u + x\frac{du}{dx}\right\} = 0, \text{ or } 2udu - 2udx - 2xdu = 0,$$

which equation involves in the same manner the ratio of the differentials of u and x.

Ex. 9. If $f(x)\,\phi(u) = au$, and $\therefore f(x')\,\phi(u') = au'$, we shall have $f(x')\,\phi(u') - f(x)\,\phi(u) = au' - au$,

or $\phi(u')\{f(x')-f(x)\}+f(x)\{\phi(u')-\phi(u)\}=a(u'-u)$;

$$\therefore\ \phi(u')\left\{\frac{\Delta f(x)}{\Delta x}\right\}+f(x)\left\{\frac{\Delta\phi(u)}{\Delta x}\right\}=a\frac{\Delta u}{\Delta x},$$

which represents the relation between the contemporaneous increments or differences of u, $\phi(u)$, x and $f(x)$; and the limits being taken, we obtain

$$\phi(u)\frac{df(x)}{dx}+f(x)\frac{d\phi(u)}{dx}=a\frac{du}{dx},$$

$$\text{or}\quad \phi(u)\,df(x)+f(x)\,d\phi(u)=a\,du,$$

which involves in the same manner the relation between their differentials.

19. In the preceding examples we have seen that the ratio of the simultaneous increments of the function u and the principal variable x admits of a limit, which has been called the Differential Coefficient because it represents the quantity by which dx is multiplied in order to obtain du: and this limit it is the immediate object of the Differential Calculus to determine, the operation performed being styled *Differentiation*. It is also manifest that the differentials of all functions whatever might be deduced by similar processes, but in very complicated expressions the operation would in general be exceedingly tedious, and on this account, certain rules are usually laid down which apply to all functions of a similar description, and which it shall be the principal object of some of the succeeding chapters to investigate.

CHAP. II.

On the Differentiation of Algebraical Functions of one principal Variable.

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20. *If* u *and* v *be two functions of the same independent variable* x, *such that for every value that can be assigned to it,* $u = v$, *then will* $\frac{du}{dx} = \frac{dv}{dx}$ *and* $du = dv$.

For, let u and v become u' and v' respectively when x has assumed the indeterminate increment h or become $x + h$; then since $u = v$ and $u' = v'$, we have immediately

$$u' - u = v' - v \text{ and } \therefore \frac{u' - u}{h} = \frac{v' - v}{h},$$

which may be written $\frac{\Delta u}{\Delta x} = \frac{\Delta v}{\Delta x}$: hence taking the limits of both sides of this equation, observing that the limits of the ratios of the finite differences are the ratios of the differentials, we obtain $\frac{du}{dx} = \frac{dv}{dx}$, and therefore $du = dv$, in both sides of which the operation of differentiation is supposed to be performed with reference to the same principal variable x.

Hence if two functions of the same principal variable be equal to one another, their differential coefficients and differentials are also equal to one another.

21. Cor. 1. If $u = v \pm a$, we shall have likewise $u' = v' \pm a$, whence $\frac{u' - u}{h} = \frac{v' - v}{h}$, and therefore as before

$$\frac{du}{dx} = \frac{dv}{dx}, \text{ and } du = dv.$$

Hence two functions of the same principal variable whose difference is an invariable quantity, have equal differentials; or, in other words, constant quantities connected to variable by the operations of *Addition* or *Subtraction* disappear by the process of differentiation.

22. Cor. 2. This is not the case however when constant and variable quantities are connected by other operations: for let $u = \frac{a}{b} v$, and in consequence of x becoming $x + h$, let u and v become u' and v' respectively, such that $u' = \frac{a}{b} v'$,

$$\therefore \text{ we have } \frac{u' - u}{h} = \frac{a}{b}\left(\frac{v' - v}{h}\right),$$

so that, if the limits of both sides be taken, there results

$$\frac{du}{dx} = \frac{a}{b}\frac{dv}{dx}, \text{ and } du = \frac{a}{b} dv.$$

Whence we conclude that all constant quantities in a function, connected with the variable by the operations of *Multiplication* or *Division*, remain unchanged when the differentials are taken.

23. *If we have* $u = p - q + r - \&c.$, *wherein* p, q, r, &c. *are all functions of the same independent variable* x, *then will*

$$\frac{du}{dx} = \frac{dp}{dx} - \frac{dq}{dx} + \frac{dr}{dx} - \&c.$$

$$and\ du = dp - dq + dr - \&c.$$

For, in consequence of x being changed into $x+h$, let p, q, r, &c. be changed into $p+i$, $q+k$, $r+l$, &c. respectively;

$$\therefore\ u' = (p+i) - (q+k) + (r+l) - \text{\&c.}$$

$$= p - q + r - \text{\&c.} + i - k + l - \text{\&c.}$$

$$\therefore\ \frac{u'-u}{h} = \frac{i-k+l-\text{\&c.}}{h} = \frac{i}{h} - \frac{k}{h} + \frac{l}{h} - \text{\&c.}:$$

whence, if we take the limits of both sides paying due regard to the definitions, we obtain

$$\frac{du}{dx} = \frac{dp}{dx} - \frac{dq}{dx} + \frac{dr}{dx} - \text{\&c.}$$

$$\text{or } du = dp - dq + dr - \text{\&c.},$$

the differentiation of each quantity being effected with reference to the change of the same principal variable x.

Whence the differential of a function composed of several terms connected by addition or subtraction, is obtained by taking the differential of every term separately, and retaining its proper sign.

24. *If we have* u = pq, p *and* q *being functions of* x, *then will*

$$\frac{\mathrm{du}}{\mathrm{dx}} = \mathrm{q}\frac{\mathrm{dp}}{\mathrm{dx}} + \mathrm{p}\frac{\mathrm{dq}}{\mathrm{dx}},\ \textit{and}\ \mathrm{du} = \mathrm{qdp} + \mathrm{pdq}.$$

For, retaining the notation of the last article, we have

$$u' = (p+i)(q+k) = pq + qi + (p+i)k,$$

$$\therefore\ \frac{u'-u}{h} = q\frac{i}{h} + (p+i)\frac{k}{h};$$

wherefore if we take the limits of both sides of this equation, observing that $p+i$ becomes ultimately equal to p, we obtain

$$\frac{du}{dx} = q\frac{dp}{dx} + p\frac{dq}{dx},\ \text{ and } \therefore\ du = qdp + pdq.$$

Hence the differential of the product of two functions of the same independent variable is found by multiplying each function by the differential of the other and adding together the two results.

25. Cor. 1. From the last article may be deduced the differential of the product of any number of factors whatever.

For, if $u = pqr$, and pq be considered as one factor, we have

$$\begin{aligned} du &= rd\,(pq) + pqdr \\ &= r(qdp + pdq) + pqdr \\ &= qrdp + prdq + pqdr: \end{aligned}$$

similarly if $u = pqrs$, we shall have

$$\begin{aligned} du &= rsd\,(pq) + pqd\,(rs) \\ &= rs\,(qdp + pdq) + pq\,(sdr + rds) \\ &= qrsdp + prsdq + pqsdr + pqrds: \end{aligned}$$

and generally from $u = pqrst$ &c., may be similarly derived

$$\begin{aligned} du = qrst \text{ \&c. } dp + prst \text{ \&c. } dq + pqst \text{ \&c. } dr \\ + pqrt \text{ \&c. } ds + pqrs \text{ \&c. } dt + \text{\&c.} \end{aligned}$$

Wherefore the differential of a continued product of functions is equal to the sum of the products formed by multiplying the differential of each of the factors by all the rest.

26. Cor. 2. From the equation $u = pq$, there has been deduced above, the result $du = qdp + pdq$:

$$\therefore \frac{du}{u} = \frac{qdp + pdq}{pq} = \frac{dp}{p} + \frac{dq}{q};$$

and similarly may it be shewn that if $u = pqrst$ &c., then will

$$\frac{du}{u} = \frac{dp}{p} + \frac{dq}{q} + \frac{dr}{r} + \frac{ds}{s} + \frac{dt}{t} + \text{\&c.}$$

27. *If* $u = \frac{p}{q}$, *where* p *and* q *are both functions of* x, *then will*

$$\frac{du}{dx} = \frac{1}{q^2}\left\{q\frac{dp}{dx} - p\frac{dq}{dx}\right\}, \text{ and } du = \frac{qdp - pdq}{q^2}.$$

For, as before $u' - u = \frac{p+i}{q+k} - \frac{p}{q}$

$$= \frac{pq + qi - pq - pk}{(q+k)\,q} = \frac{qi - pk}{(q+k)\,q},$$

$$\therefore \frac{u'-u}{h} = \frac{1}{q\,(q+k)}\left\{q\frac{i}{h} - p\frac{k}{h}\right\};$$

and the limits of both sides being taken by putting for the ratios of the increments the ratios of the differentials, and q for $q+k$, we obtain

$$\frac{du}{dx} = \frac{1}{q^2}\left\{q\frac{dp}{dx} - p\frac{dq}{dx}\right\}, \text{ and } \therefore du = \frac{qdp - pdq}{q^2}.$$

Therefore the differential of a fractional function is found by subtracting the differential of the denominator multiplied by the numerator from the differential of the numerator multiplied by the denominator, and dividing by the square of the denominator.

28. The result last obtained might have easily been deduced from that contained in (24).

For, since $u = \frac{p}{q}$, we have $uq = p$:

$$\therefore udq + qdu = dp \text{ and } qdu = dp - udq;$$

whence we obtain

$$du = \frac{dp}{q} - \frac{udq}{q} = \frac{dp}{q} - \frac{pdq}{q^2} = \frac{qdp - pdq}{q^2},$$

as before.

29. Cor. 1. If p be a constant quantity and equal to a, we have $d\left(\frac{a}{q}\right) = -\frac{a\,dq}{q^2}$; but if q be invariable and equal to b, we obtain $d\left(\frac{p}{b}\right) = \frac{dp}{b}$, as it ought by (22).

30. Cor. 2. In the same manner as in (26), if we have $u = \frac{pqr \text{ \&c.}}{p'q'r' \text{ \&c.}}$ we shall obtain by differentiation and division,

$$\frac{du}{u} = \frac{dp}{p} + \frac{dq}{q} + \frac{dr}{r} + \text{\&c.} - \frac{dp'}{p'} - \frac{dq'}{q'} - \frac{dr'}{r'} - \text{\&c.}$$

31. *If* $\mathrm{u} = \mathrm{p}^{\mathrm{m}}$, *where* p *is any function whatever of* x, *then will*

$$\frac{\mathrm{du}}{\mathrm{dx}} = \mathrm{mp}^{\mathrm{m}-1}\frac{\mathrm{dp}}{\mathrm{dx}}, \textit{ and } \mathrm{du} = \mathrm{mp}^{\mathrm{m}-1}\mathrm{dp},$$

whether m *be positive or negative, integral or fractional.*

First, let the index be a positive integer m, then retaining the notation hitherto used, we have $u' = (p + i)^m$

$$= p^m + mp^{m-1}i + \frac{m(m-1)}{1.2}p^{m-2}i^2 + \text{\&c., by the binomial}$$

theorem:

$$\therefore \frac{u' - u}{h} = \left\{mp^{m-1} + \frac{m(m-1)}{1.2}p^{m-2}i + \text{\&c.}\right\}\frac{i}{h};$$

and taking the limits of both sides of the equation, observing that all the terms after the first then become $= 0$, we get

$$\frac{du}{dx} = mp^{m-1}\frac{dp}{dx}, \text{ and } \therefore du = mp^{m-1}dp.$$

Next, let the index be a negative whole number $-m$, so that $u = p^{-m} = \frac{1}{p^m}$, then by (29) we have

$$du = -\frac{d(p^m)}{p^{2m}} = -\frac{mp^{m-1}\,dp}{p^{2m}} = -mp^{-m-1}\,dp.$$

Lastly, let the index be a positive or negative fraction represented by $\pm\frac{m}{n}$, such that $u = p^{\pm\frac{m}{n}}$, then will $u^n = p^{\pm m}$, and $\therefore\ nu^{n-1}\,du = \pm\, mp^{\pm m-1}\,dp$, by the preceding cases:

$$\therefore\ du = \pm\frac{mp^{\pm m-1}\,dp}{nu^{n-1}} = \pm\frac{m}{n}\frac{p^{\pm m-1}\,dp}{p^{\pm\frac{m(n-1)}{n}}} = \pm\frac{m}{n}p^{\pm\frac{m}{n}-1}\,dp.$$

Therefore the differential of any power or root whatever of a function is found by multiplying by the index, diminishing the index by unity and multiplying the result by the differential of the function itself.

32. The first case of the last article is manifestly only a particular case of the general formula of (25).

For, by making $p = q = r = \&c.$ to m factors,

we have $u = p^m$, and $\frac{du}{u} = m\,\frac{dp}{p}$, by (26),

whence we obtain

$$du = mu\,\frac{dp}{p} = mp^m\,\frac{dp}{p} = mp^{m-1}\,dp, \text{ as before.}$$

33. The rules which are the results of the investigations contained in the preceding articles of this chapter, will enable us to differentiate all algebraical functions of one independent variable whether explicit or implicit however complicated they may be, because all the combinations of quantities that can be formed by the common operations of Algebra have been considered in the course of their demonstrations; and though the investigations of these rules have been made by means of functions which are explicit in their form, a little consideration will readily shew us that both explicit and implicit functions

are equally included therein: and we will now proceed to apply them to the differentiation of a few functions, the operation in each case being given in full.

Ex. 1. Let $u = 4ax^3 - 5bx^{\frac{2}{5}} + 7cx^{-\frac{1}{2}} - 9x^{-4}$,

then we shall have

$$du = d\,(4ax^3 - 5bx^{\frac{2}{5}} + 7cx^{-\frac{1}{2}} - 9x^{-4}), \text{ by (20)}$$

$$= d\,(4ax^3) - d\,(5bx^{\frac{2}{5}}) + d\,(7cx^{-\frac{1}{2}}) - d\,(9x^{-4}), \text{ by (23)}$$

$$= 4ad\,(x^3) - 5bd\,(x^{\frac{2}{5}}) + 7cd\,(x^{-\frac{1}{2}}) - 9d\,(x^{-4}), \text{ by (22)}$$

$$= 4a.3x^2dx - 5b.\frac{2}{5}x^{-\frac{3}{5}}dx + 7c\times -\tfrac{1}{2}x^{-\frac{3}{2}}dx - 9\times -4x^{-5}dx,$$

by (31)

$$= 12ax^2dx - 2bx^{-\frac{3}{5}}dx - \frac{7c}{2}x^{-\frac{3}{2}}dx + 36x^{-5}dx,$$

by reduction: and the differential of the proposed function being thus determined according to the rules, we obtain immediately the differential coefficient

$$\frac{du}{dx} = 12ax^2 - 2bx^{-\frac{3}{5}} - \frac{7c}{2}x^{-\frac{3}{2}} + 36x^{-5},$$

which is a new function dependent upon the same principal variable x.

Ex. 2. Let the function proposed be

$u = (a^2 + x^2)^2 - (b^2 - x^2)^2$, then as before, we get

$$du = d\,\{(a^2 + x^2)^2\} - d\,\{(b^2 - x^2)^2\}, \text{ by (20) and (23)}$$

$$= 2\,(a^2 + x^2)\,d\,(a^2 + x^2) - 2\,(b^2 - x^2)\,d\,(b^2 - x^2), \text{ by (31)}$$

$$= 2\,(a^2 + x^2)\,d\,(x^2) - 2\,(b^2 - x^2)\,d\,(-x^2), \text{ by (21)}$$

$$= 2\,(a^2 + x^2)\,2xdx + 2\,(b^2 - x^2)\,2xdx, \text{ by (31)}$$

$$= 4\,(a^2 + b^2)\,xdx, \text{ by reduction;}$$

whence we shall obviously have the differential coefficient $\frac{du}{dx} = 4(a^2 + b^2)\,x$, which is a new function of x.

Ex. 3. Let the proposed function be in the form of a product of two quantities as $u = (1 + x)(1 + x^2)$, then will

$$\begin{aligned} du &= (1 + x^2)\,d(1 + x) + (1 + x)\,d(1 + x^2), \text{ by (20) and (24)} \\ &= (1 + x^2)\,dx + (1 + x)\,2x\,dx, \text{ by (21) and (31)} \\ &= (1 + 2x + 3x^2)\,dx, \text{ by reduction:} \end{aligned}$$

whence the differential coefficient $\frac{du}{dx} = 1 + 2x + 3x^2$, a new function of x derived from the one proposed by differentiation.

Ex. 4. If the function be the continued product of three factors as $u = x(a + x)(b + 2x)$, we have by (20) and (25)

$$\begin{aligned} du &= (a+x)(b+2x)\,dx + x(b+2x)\,d(a+x) + x(a+x)\,d(b+2x) \\ &= (a+x)(b+2x)\,dx + x(b+2x)\,dx + x(a+x)\,2dx, \text{ by (21)} \\ &= \{ab + (4a + 2b)\,x + 6x^2\}\,dx: \end{aligned}$$

$$\text{and therefore } \frac{du}{dx} = ab + (4a + 2b)\,x + 6x^2.$$

Ex. 5. Let us take the fractional function $u = \frac{ax}{a^2 + x^2}$;

$$\begin{aligned} \therefore du &= \frac{(a^2 + x^2)\,d(ax) - ax\,d(a^2 + x^2)}{(a^2 + x^2)^2}, \text{ by (20) and (27)} \\ &= \frac{(a^2 + x^2)\,a\,dx - 2ax^2dx}{(a^2 + x^2)^2}, \text{ by (22) and (31)} \\ &= \frac{(a^3 - ax^2)\,dx}{(a^2 + x^2)^2} = \frac{a^2 - x^2}{(a^2 + x^2)^2}\,a\,dx, \text{ by reduction:} \end{aligned}$$

whence as before the differential coefficient $\frac{du}{dx} = \frac{a(a^2 - x^2)}{(a^2 + x^2)^2}$.

Ex. 6. If there be proposed $u = \frac{a + 2bx}{(a+bx)^2}$, we shall have

$$du = \frac{d(a+2bx)}{(a+bx)^2} - \frac{(a+2bx)\,d(a+bx)^2}{(a+bx)^4}, \text{ by (28)}$$

$$= \frac{2b\,dx}{(a+bx)^2} - \frac{(a+2bx)\,2(a+bx)\,b\,dx}{(a+bx)^4}, \text{ by (21) and (31)}$$

$$= \frac{2b(a+bx)^2 dx - (a+2bx)\,2b\,(a+bx)\,dx}{(a+bx)^4}$$

$$= \frac{2b(a+bx) - 2b(a+2bx)}{(a+bx)^3}\,dx = -\frac{2b^2x\,dx}{(a+bx)^3},$$

by reduction: and thence it is manifest that the differential coefficient

$$\frac{du}{dx} = -\frac{2b^2x}{(a+bx)^3}.$$

Ex. 7. Let the function proposed be irrational as

$$u = (2a^2 + 3x^2)(a^2 - x^2)^{\frac{3}{2}}, \text{ then by (24) and (31),}$$

$$du = (a^2 - x^2)^{\frac{3}{2}}\,6x\,dx - (2a^2 + 3x^2)\,\tfrac{3}{2}\,(a^2 - x^2)^{\frac{1}{2}}\,2x\,dx$$

$$= 6x\,dx(a^2 - x^2)^{\frac{1}{2}}(a^2 - x^2) - 3x\,dx(a^2 - x^2)^{\frac{1}{2}}(2a^2 + 3x^2)$$

$$= -15x^3\sqrt{a^2 - x^2}\,dx, \text{ by reduction;}$$

and thus we obtain the differential coefficient

$$\frac{du}{dx} = -15x^3\sqrt{a^2 - x^2}.$$

Ex. 8. Let the function be an irrational fraction as

$$u = \frac{\sqrt[3]{1+x^2}}{\sqrt{1+x}} = \frac{(1+x^2)^{\frac{1}{3}}}{(1+x)^{\frac{1}{2}}}, \text{ then by (27) we have}$$

$$du = \frac{(1+x)^{\frac{1}{2}}\, d\,(1+x^2)^{\frac{1}{3}} - (1+x^2)^{\frac{1}{3}}\, d\,(1+x)^{\frac{1}{2}}}{1+x}$$

$$= \frac{(1+x)^{\frac{1}{2}}\,\frac{1}{3}(1+x^2)^{-\frac{2}{3}}\,d\,(1+x^2) - (1+x^2)^{\frac{1}{3}}\,\frac{1}{2}(1+x)^{-\frac{1}{2}}\,d\,(1+x)}{1+x}$$

$$= \frac{\frac{2}{3}x(1+x)\,dx - \frac{1}{2}(1+x^2)\,dx}{(1+x)^{\frac{3}{2}}(1+x^2)^{\frac{2}{3}}} = \frac{(x^2+4x-3)\,dx}{6(1+x)^{\frac{3}{2}}(1+x^2)^{\frac{2}{3}}}, \text{ by reduction;}$$

and the differential coefficient

$$\frac{du}{dx} = \frac{x^2+4x-3}{6\,(1+x)^{\frac{3}{2}}\,(1+x^2)^{\frac{2}{3}}},$$

which is also of the form $u = f(x)$.

Ex. 9. If we have an implicit function proposed as $u^2 - 2u\sqrt{1+x^2} + x^2 = 0$, then by (24) and (31) we get

$$2\,udu - 2\sqrt{1+x^2}\,du - \frac{2\,uxdx}{\sqrt{1+x^2}} + 2\,xdx = 0,$$

$$\text{or } \{u - \sqrt{1+x^2}\}\,du = \left\{\frac{u - \sqrt{1+x^2}}{\sqrt{1+x^2}}\right\} xdx:$$

and $\therefore$ the differential coefficient $\dfrac{du}{dx} = \dfrac{x}{\sqrt{1+x^2}}$, which is the same as would have been obtained by first expressing u in terms of x, and then differentiating the function with respect to it.

Ex. 10. Let $u = \sqrt{ax + \sqrt{bx + \sqrt{ax + \&c.\ in\ infinitum}}}$; then establishing the relation between u and x in finite terms, we obtain

$$u^4 - 2\,au^2x - u + a^2x^2 - bx = 0:$$

$$\therefore\ 4u^3du - 4auxdu - 2au^2dx - du + 2a^2xdx - bdx = 0,$$

$$\text{or}\ (4u^3 - 4aux - 1)\,du - (2au^2 - 2a^2x + b)\,dx = 0\,;$$

whence we obtain the differential coefficient

$$\frac{du}{dx} = \frac{2au^2 - 2a^2x + b}{4u^3 - 4aux - 1},$$

wherein u is an implicit function of x.

Ex. 11. If $u = \dfrac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}}$, the operation of differentiation will be much simplified by multiplying the numerator and denominator of the fraction by the numerator, which gives $u = \dfrac{a + \sqrt{a^2 - x^2}}{x}$, whence by (28) we have

$$du = \frac{1}{x} \times -\frac{xdx}{\sqrt{a^2 - x^2}} - (a + \sqrt{a^2 - x^2})\frac{dx}{x^2}$$

$$= -\frac{dx}{\sqrt{a^2 - x^2}} - \frac{(a + \sqrt{a^2 - x^2})dx}{x^2} = -\frac{a^2 + a\sqrt{a^2 - x^2}}{x^2\sqrt{a^2 - x^2}}\,dx,$$

by reduction: whence as before, we have

$$\frac{du}{dx} = -\frac{a^2 + a\sqrt{a^2 - x^2}}{x^2\sqrt{a^2 - x^2}},\ \text{the differential coefficient.}$$

Ex. 12. Let $u = \dfrac{x}{x + \sqrt{1 - x^2}}$, then by means of (30),

$$\frac{du}{u} = \frac{dx}{x} - \frac{d\,(x + \sqrt{1 - x^2})}{x + \sqrt{1 - x^2}} = \frac{dx}{x} - \frac{dx - \dfrac{xdx}{\sqrt{1 - x^2}}}{x + \sqrt{1 - x^2}}$$

$$= \frac{dx}{x} + \frac{x - \sqrt{1 - x^2}}{x + \sqrt{1 - x^2}}\,\frac{dx}{\sqrt{1 - x^2}} = \frac{1}{(x + \sqrt{1 - x^2})x\sqrt{1 - x^2}},$$

$$\therefore du = \frac{u\,dx}{(x+\sqrt{1-x^2})\,x\sqrt{1-x^2}} = \frac{dx}{\sqrt{1-x^2}\,(x+\sqrt{1-x^2})^2}:$$

and thence we obtain the differential coefficient

$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}\,(x+\sqrt{1-x^2})^2}.$$

Ex. 13. Lastly, let $u = \sqrt[4]{\left\{a - \frac{b}{\sqrt{x}} + \sqrt[3]{(c^2-x^2)^2}\right\}^3}$;

then if we put $p = \frac{b}{\sqrt{x}}$ and $q = \sqrt[3]{(c^2-x^2)^2}$, we shall have

$$u = \sqrt[4]{(a-p+q)^3} = (a-p+q)^{\frac{3}{4}}:$$

$$\therefore du = \tfrac{3}{4}(a-p+q)^{-\frac{1}{4}}(-dp+dq) = \frac{-3dp+3dq}{4\sqrt[4]{a-p+q}};$$

$$\text{but } dp = \frac{-b\,dx}{2x\sqrt{x}} \text{ and } dq = \frac{-4x\,dx}{3\sqrt[3]{c^2-x^2}},$$

whence by substitution and reduction we obtain

$$du = \frac{3b\sqrt[3]{c^2-x^2} - 8x^2\sqrt{x}}{4\sqrt[4]{a - \frac{b}{\sqrt{x}} + \sqrt[3]{(c^2-x^2)^2}}}\;\frac{dx}{2x^{\frac{3}{2}}\sqrt[3]{c^2-x^2}}.$$

Other artifices by which the differentiation of functions may be greatly facilitated, will frequently suggest themselves, but as no general rules can be laid down, the reader is recommended to acquire a dexterity of operation by practising the examples appended to the end of the work.

CHAP. III.

On the Differentiation of Exponential and Logarithmical Functions of one principal Variable.

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34. *If* $u = a^p$, *where* p *is any function whatever of* x, *then will* $du = \log a a^p dp$, *where the logarithm of* a *is taken in the system whose base is* 2.71828 &c. *represented by* e.

For, retaining the notation used in the last chapter, we have $u = a^p$ and $u' = a^{p+i}$;

$$\therefore u' - u = a^{p+i} - a^p = a^p(a^i - 1), \text{ and } \frac{u' - u}{h} = a^p\left(\frac{a^i - 1}{h}\right):$$

but $a^i = \{1 + (a-1)\}^i = 1 + i(a-1) + \dfrac{i(i-1)}{1.2}(a-1)^2$

$+ \dfrac{i(i-1)(i-2)}{1.2.3}(a-1)^3 + \&c.$ by the binomial theorem:

$$\therefore \frac{u' - u}{h}$$

$$= a^p\left\{(a-1) + (a-1)^2\frac{(i-1)}{1.2} + (a-1)^3\frac{(i-1)(i-2)}{1.2.3} + \&c.\right\}\frac{i}{h},$$

which represents the relation between the finite differences of u and x: hence, taking the limits of both sides by diminishing indefinitely the corresponding increments, we get

$$\frac{du}{dx} = a^p \left\{(a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \&c.\right\} \frac{dp}{dx} = ka^p \frac{dp}{dx},$$

if k be taken to represent the quantity

$$\left\{(a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \&c.\right\};$$

but it has been proved in Algebra, that $e^k = a$; $\therefore k = \log a$, in the system whose base is e; whence we obtain

$$\frac{du}{dx} = \log a a^p \frac{dp}{dx}, \text{ and } \therefore du = \log a a^p dp.$$

Hence the differential of an exponential function whose index alone is variable, is equal to the continued product of the *Naperian* logarithm of the root, the function itself and the differential of its index.

35. Had the algebraical expansion of a^p been assumed, its differential might have been obtained, by the immediate application of some of the rules investigated in the preceding chapter.

For, since $u = a^p = 1 + kp + \dfrac{k^2p^2}{1.2} + \dfrac{k^3p^3}{1.2.3} + \dfrac{k^4p^4}{1.2.3.4} + \&c.$, we shall have by (20) and (31)

$$du = kdp + k^2pdp + \frac{k^3p^2dp}{1.2} + \frac{k^4p^3dp}{1.2.3} + \&c.$$

$$= \left\{1 + kp + \frac{k^2p^2}{1.2} + \frac{k^3p^3}{1.2.3} + \&c.\right\} kdp = ka^p dp, \text{ as before.}$$

36. Cor. 1. We have above assumed $k = \log a$, but if we suppose the logarithms of a and e to be taken in any other system, we shall have

$$k = \frac{\text{Log } a}{\text{Log } e}, \text{ and } \therefore du = \frac{\text{Log } a}{\text{Log } e} a^p dp:$$

but since in analysis, the *Naperian* logarithms whose base is e are generally made use of, we shall always suppose the loga-

rithms to be taken in that system unless the contrary be expressed and shall always distinguish the different systems by the small and large initials as above.

37. Cor. 2. Hence if $u = e^p$, we shall obviously have $du = \log e e^p dp = e^p dp$.

38. Cor. 3. If the function proposed be $u = e^{e^x}$, then will $du = e^{e^x} e^x dx$.

For, putting $p = e^x$, we have $u = e^p$ and therefore

$$du = d\,(e^p) = e^p dp = e^p d\,(e^x) = e^p e^x dx = e^{e^x} e^x dx.$$

Similarly, if $u = e^{e^{e^x}}$, we shall have $du = e^{e^{e^x}} e^{e^x} e^x dx$: and the differential will be of the same form whatever number of times the quantity e is repeated.

39. *If* u = *Log* p, *where* p *is any function of* x, *then will* du $= \mathrm{M}\dfrac{\mathrm{dp}}{\mathrm{p}}$, *wherein* M *is the modulus of the system of logarithms used.*

For, let a be the base of the system in which the logarithm is supposed to be taken, then since $u = \text{Log}\, p$, we have by the nature of logarithms $a^u = p$; whence by (34),

$$\log a\, a^u du = dp, \text{ and } \therefore\ du = \frac{1}{\log a}\,\frac{dp}{a^u} = \frac{1}{\log a}\,\frac{dp}{p}:$$

therefore if the multiplier $\dfrac{1}{\log a}$, which connects the system whose base is a with the *Naperian* system and is called the *Modulus* of the former system, be denoted by M, we shall have $du = M\dfrac{dp}{p}$.

Whence the differential of the logarithm of any function is equal to the modulus of the system of logarithms used, multiplied by the differential of the function and divided by the function itself.

40. If the Logarithm of p were expressed algebraically in terms of p as

$$u = \text{Log}p = M\{(p-1) - \tfrac{1}{2}(p-1)^2 + \tfrac{1}{3}(p-1)^3 - \text{\&c. } \textit{in infinitum}\},$$

we should have immediately

$$du = M\{dp - (p-1)\,dp + (p-1)^2\,dp - \text{\&c. } \textit{in infinitum}\}$$

$$= M\left(\frac{1}{1+p-1}\right)dp = M\frac{dp}{p}, \text{ as above found.}$$

41. Cor. Hence in the *Naperian* system where $u = \log p$, we shall have $du = \dfrac{dp}{p}$.

Therefore the differential of the *Naperian* logarithm of a function, is equal to the differential of the function divided by the function itself.

42. *If* $u = p^q$ *where* p *and* q *are both functions of the same independent variable* x, *then will*

$$du = p^q \log p\,dq + qp^{q-1}dp.$$

For, since $u = p^q$ we have $\log u = q \log p$,

$$\therefore \frac{du}{u} = \log p\,dq + q\frac{dp}{p}, \text{ by preceding articles:}$$

$$\text{whence } du = u \log p\,dq + uq\frac{dp}{p} = p^q \log p\,dq + qp^{q-1}dp.$$

Hence the differential of an exponential function in which both the root and index are variable, is equal to the sum of the differentials found by considering each separately as constant and the other variable.

43. Cor. If $u = p^{q^r}$, p, q and r being all functions of the same principal variable x, we shall have $\log u = q^r \log p$:

$$\therefore \frac{du}{u} = \log p d(q^r) + q^r d(\log p)$$

$$= \log p \{q^r \log q dr + r q^{r-1} dq\} + q^r \frac{dp}{p}, \text{ by (42) and (41):}$$

$$\text{whence } du = u q^r \log p \log q dr + u \log p r q^{r-1} dq + u q^r \frac{dp}{p}$$

$$= p^{q^r} q^r \left\{\frac{dp}{p} + r \log p \frac{dq}{q} + r \log p \log q \frac{dr}{r}\right\}:$$

and similar methods may be applied to other functions of a like description.

44. *If* u = *log of log* p, *which is usually written* u = *log*² p, *then will* $\mathrm{du} = \frac{\mathrm{dp}}{\mathrm{p} \log \mathrm{p}}$.

For, if $q = \log p$, we shall have $u = \log q$, and $\therefore du = \frac{dq}{q}$:

but $dq = d(\log p) = \frac{dp}{p}$, whence $du = \frac{dp}{pq} = \frac{dp}{p \log p}$.

Also, if $u = \log$ of $\log$ of $\log p = \log^3 p$, similar substitutions will give us $du = \frac{dp}{p \log p \log^2 p}$: and generally if we have a logarithmic function of the n^{th} order, as $u = \log^n p$, we should obtain

$$du = \frac{dp}{p \log p \log^2 p \log^3 p \text{ \&c. } \log^{n-1} p}.$$

45. We shall now illustrate the use of the formulæ just investigated, which comprehend the differentiation of all kinds of exponential and logarithmical functions whatever, by their application to a few examples.

Ex. 1. Let the proposed function be in the form of an exponential fraction as $u = \frac{a^x - 1}{a^x + 1}$, then we get

$$du = \frac{(a^x+1)\,d\,(a^x-1) - (a^x-1)\,d\,(a^x+1)}{(a^x+1)^2}, \text{ by (27)}$$

$$= \frac{(a^x+1)\log a a^x dx - (a^x-1)\log a a^x dx}{(a^x+1)^2}, \text{ by (34)}$$

$$= 2\log a \frac{a^x dx}{(a^x+1)^2}, \text{ by reduction:}$$

and the differential coefficient $\frac{du}{dx} = \frac{2\log a a^x}{(a^x+1)^2}$.

Ex. 2. If the function proposed be logarithmical as

$u = \text{Log}\left(\frac{a+x}{a-x}\right) = \text{Log}\,(a+x) - \text{Log}\,(a-x)$, we shall have

$$du = M\frac{d\,(a+x)}{a+x} - M\frac{d\,(a-x)}{a-x}, \text{ by (39)}$$

$$= M\left\{\frac{dx}{a+x} + \frac{dx}{a-x}\right\} = M\frac{2\,a\,dx}{a^2 - x^2}:$$

and thence $\frac{du}{dx} = M\frac{2\,a}{a^2-x^2}$, which is a new function of x.

Ex. 3. If $u = \log\left\{2x+1+2\sqrt{1+x+x^2}\right\}$, we shall have

$$du = \frac{d\,(2x+1+2\sqrt{1+x+x^2})}{2x+1+2\sqrt{1+x+x^2}}, \text{ by (41)}$$

$$= \frac{2\,dx + (1+x+x^2)^{-\frac{1}{2}}(dx + 2\,x\,dx)}{2\,x+1+2\sqrt{1+x+x^2}}, \text{ by (31)}$$

$$= \frac{2\sqrt{1+x+x^2}+1+2\,x}{2\,x+1+2\sqrt{1+x+x^2}}\,\frac{dx}{\sqrt{1+x+x^2}} = \frac{dx}{\sqrt{1+x+x^2}}:$$

and the differential coefficient $\dfrac{du}{dx} = \dfrac{1}{\sqrt{1+x+x^2}}$.

Ex. 4. Let the function proposed be $u = \dfrac{e^x}{1+x}$, then from (27) we obtain

$$du = \frac{(1+x)\,d(e^x) - e^x d(1+x)}{(1+x)^2} = \frac{(1+x)\,e^x dx - e^x dx}{(1+x)^2}, \text{ by (37)}$$

$$= \frac{e^x x\,dx}{(1+x)^2}: \text{ and therefore } \frac{du}{dx} = \frac{e^x x}{(1+x)^2}.$$

Ex. 5. Let $u = \left(\dfrac{a}{x}\right)^x$, then we shall have

$$du = \log\left(\frac{a}{x}\right)\left(\frac{a}{x}\right)^x dx + x\left(\frac{a}{x}\right)^{x-1} d\left(\frac{a}{x}\right), \text{ by (42)}$$

$$= \log\left(\frac{a}{x}\right)\left(\frac{a}{x}\right)^x dx - a\left(\frac{a}{x}\right)^{x-1}\frac{dx}{x}$$

$$= \left\{\log\left(\frac{a}{x}\right) - 1\right\}\left(\frac{a}{x}\right)^x dx: \text{ and } \frac{du}{dx} = \left\{\log\left(\frac{a}{x}\right) - 1\right\}\left(\frac{a}{x}\right)^x.$$

46. In the same manner as in some of the preceding articles, the introduction of logarithms will in many algebraical functions which are much complicated, greatly facilitate the operation of differentiation.

Ex. 1. If $u = x^{\frac{3}{2}}\sqrt{\dfrac{1+x}{1-x}}$, by taking the *Naperian* logarithms of both sides, we obtain

$$\log u = \tfrac{3}{2}\log x + \tfrac{1}{2}\log(1+x) - \tfrac{1}{2}\log(1-x);$$

$$\therefore \frac{du}{u} = \frac{3}{2}\frac{dx}{x} + \frac{1}{2}\frac{dx}{1+x} + \frac{1}{2}\frac{dx}{1-x}, \text{ by (41)}$$

$$= \frac{3+2x-3x^2}{2x(1-x^2)}\,dx, \text{ by reduction:}$$

$$\text{whence } du = x^{\frac{3}{2}}\sqrt{\frac{1+x}{1-x}}\,\frac{(3+2x-3x^2)\,dx}{2x(1-x^2)}$$

$$= \frac{3+2x-3x^2}{2(1-x)}\sqrt{\frac{x}{1-x^2}}\,dx.$$

Ex. 2. Let $u = \dfrac{p^m q^n \text{ \&c.}}{p'^{m'} q'^{n'} \text{ \&c.}}$, and we have similarly

$$\log u = m\log p + n\log q + \text{\&c.} - m'\log p' - n'\log q' - \text{\&c.}$$

$$\therefore \frac{du}{u} = \frac{m\,dp}{p} + \frac{n\,dq}{q} + \text{\&c.} - \frac{m'\,dp'}{p'} - \frac{n'\,dq'}{q'} - \text{\&c. by (41);}$$

$$\therefore du = \frac{mu\,dp}{p} + \frac{nu\,dq}{q} + \text{\&c.} - \frac{m'u\,dp'}{p'} - \frac{n'u\,dq'}{q'} - \text{\&c.}$$

$$= \frac{mp^{m-1}q^n \text{ \&c. } dp}{p'^{m'}q'^{n'} \text{ \&c.}} + \frac{np^m q^{n-1} \text{ \&c. } dq}{p'^{m'}q'^{n'} \text{ \&c.}} + \text{\&c.}$$

$$- \frac{m'p^m q^n \text{ \&c. } dp'}{p'^{m'+1}q'^{n'} \text{ \&c.}} - \frac{n'p^m q^n \text{ \&c. } dq'}{p'^{m'}q'^{n'+1} \text{ \&c.}} - \text{\&c.}$$

CHAP. IV.

On the Differentiation of Trigonometrical and Geometrical Functions of one principal Variable.

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47. *If* u = *sin* p, *where* p *may be any function of* x *whatever, then will*

$$\frac{\mathrm{du}}{\mathrm{dx}} = \cos \mathrm{p}\frac{\mathrm{dp}}{\mathrm{dx}}, \textit{ and } \mathrm{du} = \cos \mathrm{pdp}.$$

For, the notation of the preceding chapters being still retained, we have

$$u' = \sin (p+i), \text{ and } \therefore\ u' - u = \sin (p+i) - \sin p$$

$$= 2\cos (p + \tfrac{1}{2}i) \sin \tfrac{1}{2}i, \text{ by (67) Trig.:}$$

$$\therefore\ \frac{u'-u}{h} = 2\cos (p + \tfrac{1}{2}i)\frac{\sin \frac{1}{2}i}{h} = \left\{\cos (p + \tfrac{1}{2}i)\frac{\sin \frac{1}{2}i}{\frac{1}{2}i}\right\}\frac{i}{h};$$

therefore taking the limits as before, observing that $p + \frac{1}{2}i$ is ultimately $= p$, and $\dfrac{\sin \frac{1}{2}i}{\frac{1}{2}i} = 1$ as appears from (2) of the introductory chapter, we get

$$\frac{du}{dx} = \cos p\frac{dp}{dx}, \text{ and } \therefore\ du = \cos pdp.$$

If the radius be supposed to be a instead of 1, we shall obviously have by (60) Trig.: $du = \frac{1}{a}\cos p\,dp$.

48. Cor. 1. From the formula of the last article, the rules laid down in the second chapter will enable us to deduce the differentials of all the other trigonometrical functions.

Since, $\cos p = \sin\left(\frac{\pi}{2} - p\right)$, we shall have $d\cos p$

$$= d\sin\left(\frac{\pi}{2} - p\right) = \cos\left(\frac{\pi}{2} - p\right) d\left(\frac{\pi}{2} - p\right) = -\sin p\,dp.$$

Since, $\operatorname{vers} p = 1 - \cos p$, we have $d\operatorname{vers} p = d(1 - \cos p)$ $= -d\cos p = \sin p\,dp$.

Since, $\operatorname{chd} p = 2\sin\frac{1}{2}p$, $\therefore d\operatorname{chd} p = 2d\sin\frac{1}{2}p = \cos\frac{1}{2}p\,dp$.

Since, $\tan p = \frac{\sin p}{\cos p}$, $\therefore d\tan p = \frac{\cos p\,d\sin p - \sin p\,d\cos p}{(\cos p)^2}$

$$= \frac{(\cos p)^2 + (\sin p)^2}{(\cos p)^2}dp = \frac{dp}{(\cos p)^2} = (\sec p)^2\,dp.$$

Since, $\cot p = \tan\left(\frac{\pi}{2} - p\right)$, $\therefore d\cot p = d\tan\left(\frac{\pi}{2} - p\right)$

$$= \left\{\sec\left(\frac{\pi}{2} - p\right)\right\}^2 d\left(\frac{\pi}{2} - p\right) = -(\operatorname{cosec} p)^2\,dp.$$

Since, $\sec p = \frac{1}{\cos p}$, $\therefore d\sec p = -\frac{d\cos p}{(\cos p)^2} = \frac{\sin p\,dp}{(\cos p)^2}$ $= \tan p\sec p\,dp$.

Since, $\operatorname{cosec} p = \sec\left(\frac{\pi}{2} - p\right)$, $\therefore\ d \operatorname{cosec} p = d \sec\left(\frac{\pi}{2} - p\right)$

$$= \tan\left(\frac{\pi}{2} - p\right)\sec\left(\frac{\pi}{2} - p\right) d\left(\frac{\pi}{2} - p\right) = -\cot p \operatorname{cosec} p dp.$$

49. Cor. 2. We have therefore the following results adapted to the radii 1 and a respectively.

To the radius 1.	*To the radius* a.
(1). $d \sin p = \cos p dp$:	$d \sin p = \frac{1}{a} \cos p dp$:
(2). $d \cos p = -\sin p dp$:	$d \cos p = -\frac{1}{a} \sin p dp$:
(3). $d \operatorname{vers} p = \sin p dp$:	$d \operatorname{vers} p = \frac{1}{a} \sin p dp$:
(4). $d \operatorname{chd} p = \cos \frac{1}{2} p dp$:	$d \operatorname{chd} p = \frac{1}{a} \cos \frac{1}{2} p dp$:
(5). $d \tan p = (\sec p)^2 dp$:	$d \tan p = \frac{1}{a^2} (\sec p)^2 dp$:
(6). $d \cot p = -(\operatorname{cosec} p)^2 dp$:	$d \cot p = -\frac{1}{a^2} (\operatorname{cosec} p)^2 dp$:
(7). $d \sec p = \tan p \sec p dp$:	$d \sec p = \frac{1}{a^2} \tan p \sec p dp$:
(8). $d \operatorname{cosec} p = -\cot p \operatorname{cosec} p dp$;	$d \operatorname{cosec} p = -\frac{1}{a^2} \cot p \operatorname{cosec} p dp$;

which by trigonometrical substitution may all be made to assume a variety of different forms.

50. The results contained in the last two articles might have been readily deduced by means of the algebraical or exponential expressions for the respective functions investigated in Articles (276), and (282), &c. of the Trigonometry, and indeed if the differential of any one of the Trigonometrical functions be investigated from first principles as in (47), those of all the rest may easily be derived from it.

51. By means of the preceding articles we are enabled to find immediately the differentials of the inverse trigonometrical functions $\sin^{-1}p$, $\cos^{-1}p$, &c.

If $u=\sin^{-1}p$, $\therefore\ \sin u=p$, and $\cos u\,du=dp$;

$$\therefore\ du=\frac{dp}{\cos u}=\frac{dp}{\sqrt{1-p^2}}.$$

If $u=\cos^{-1}p$, $\therefore\ \cos u=p$, and $-\sin u\,du=dp$;

$$\therefore\ du=-\frac{dp}{\sin u}=-\frac{dp}{\sqrt{1-p^2}}.$$

If $u=\text{vers}^{-1}p$, $\therefore\ \text{vers}\,u=p$, [illegible] $du=dp$;

$$\therefore\ du=\frac{dp}{\sin u}=\frac{dp}{\sqrt{2p-p^2}}.$$

If $u=\text{chd}^{-1}p$, $\therefore\ \text{chd}\,u=p$, and $\cos\frac{1}{2}u\,du=dp$;

$$\therefore\ du=\frac{dp}{\cos\frac{1}{2}u}=\frac{2dp}{\sqrt{4-p^2}}.$$

If $u=\tan^{-1}p$, $\therefore\ \tan u=p$, and $(\sec u)^2\,du=dp$;

$$\therefore\ du=\frac{dp}{(\sec u)^2}=\frac{dp}{1+p^2}.$$

If $u=\cot^{-1}p$, $\therefore\ \cot u=p$, and $-(\text{cosec}\,u)^2\,du=dp$;

$$\therefore\ du=-\frac{dp}{(\text{cosec}\,u)^2}=-\frac{dp}{1+p^2}.$$

If $u = \sec^{-1} p$ $\therefore$ $\sec u = p$, and $\tan u \sec u \, du = dp$;

$$\therefore du = \frac{dp}{\tan u \sec u} = \frac{dp}{p\sqrt{p^2-1}}.$$

If $u = \operatorname{cosec}^{-1} p$, $\therefore$ $\operatorname{cosec} u = p$, and $-\cot u \operatorname{cosec} u \, du = dp$;

$$\therefore du = -\frac{dp}{\cot u \operatorname{cosec} u} = -\frac{dp}{p\sqrt{p^2-1}}.$$

52. Cor. To the radii 1 and a respectively, we shall therefore have the following results.

To the radius 1.	*To the radius* a.
(1). $d \sin^{-1} p = \frac{dp}{\sqrt{1-p^2}}$:	$d \sin^{-1} p = \frac{adp}{\sqrt{a^2-p^2}}$:
(2). $d \cos^{-1} p = -\frac{dp}{\sqrt{1-p^2}}$:	$d \cos^{-1} p = -\frac{adp}{\sqrt{a^2-p^2}}$:
(3). $d \operatorname{vers}^{-1} p = \frac{dp}{\text{[illegible]}}$:	$d \operatorname{vers}^{-1} p = -\frac{adp}{\sqrt{2ap-p^2}}$:
(4). $d \operatorname{chd}^{-1} p = -$ [illegible] $-p^2}$:	$d \operatorname{chd}^{-1} p = \frac{2adp}{\sqrt{4a^2-p^2}}$:
(5). $d \tan^{-1} p = \frac{dp}{1+p^2}$:	$d \tan^{-1} p = \frac{a^2 dp}{a^2+p^2}$:
(6). $d \cot^{-1} p = -\frac{dp}{1+p^2}$:	$d \cot^{-1} p = -\frac{a^2 dp}{a^2+p^2}$:
(7). $d \sec^{-1} p = \frac{dp}{p\sqrt{p^2-1}}$:	$d \sec^{-1} p = \frac{a^2 dp}{p\sqrt{p^2-a^2}}$:
(8). $d \operatorname{cosec}^{-1} p = -\frac{dp}{p\sqrt{p^2-1}}$.	$d \operatorname{cosec}^{-1} p = -\frac{a^2 dp}{p\sqrt{p^2-a^2}}$.

53. Substitutions similar to those made in (44), will enable us to determine with facility the differentials of trigonometrical functions of superior orders.

If $u = \sin$ of $\sin p$, which is usually written $u = \sin^2 p$, assume $\sin p = q$, and $\therefore u = \sin q$: whence $du = d \sin q = \cos q\, dq = \cos q d \sin p = \cos q \cos p dp = \cos \sin p \cos p dp$.

Again, if $u = \sin \cos p$, we shall similarly obtain

$$du = - \cos^2 p \sin p dp.$$

Lastly, if $u = \sin \log p$, and we put $\log p = q$, we shall have

$$du = d \sin q = \cos q dq = \cos q \frac{dp}{p} = \cos \log p \frac{dp}{p}.$$

By the same mode of proceeding we shall readily prove that

$$d \sin^n p = \cos p \cos \sin p \cos \sin^2 p \text{ \&c. } \cos \sin^{n-1} p dp;$$

$$d \cos^n p = (-1)^n \sin p \sin \cos p \sin \cos^2 p \text{ \&c. } \sin \cos^{n-1} p dp.$$

54. Before we proceed further, we will illustrate the use of the formulæ just investigated by their application to a few examples.

Ex. 1. If $u = \sin x \cos 2x$, then we shall obviously have

$$\begin{aligned} du &= \cos 2x d \sin x + \sin x d \cos 2x, \text{ by (24)} \\ &= \cos 2x \cos x dx - \sin x \sin 2x\, 2dx, \text{ by (47) and (48)} \\ &= (\cos 2x \cos x - 2 \sin x \sin 2x)\, dx, \\ \text{or } &= (2 \cos 3x - \cos 2x \cos x)\, dx, \text{ by substitution.} \end{aligned}$$

Ex. 2. If $u = \dfrac{\sin nx}{(\cos x)^n}$, we shall have

$$du = \frac{(\cos x)^n d \sin nx - \sin nx d (\cos x)^n}{(\cos x)^{2n}}, \text{ by (27)}$$

$$= \frac{n (\cos x)^n \cos nx dx + n \sin nx (\cos x)^{n-1} \sin x dx}{(\cos x)^{2n}},$$

by (47), and (48)

$$= \frac{n(\cos nx \cos x + \sin nx \sin x)\,dx}{(\cos x)^{n+1}} = \frac{n\cos(n-1)\,x\,dx}{(\cos x)^{n+1}}.$$

Ex. 3. If $u = 3\tan x + (\tan x)^3$, then will

$$du = \left\{1 + (\tan x)^2\right\} 3\,d\tan x$$

$$= (\sec x)^2\, 3(\sec x)^2\, dx, \text{ by (48)}$$

$$= 3(\sec x)^4\, dx.$$

Ex. 4. If $u = \sin^{-1}(2x\sqrt{1-x^2})$, then $\sin u = 2x\sqrt{1-x^2}$;

$$\therefore \cos u\,du = 2\sqrt{1-x^2}\,dx - \frac{2x^2 dx}{\sqrt{1-x^2}} = \frac{2(1-2x^2)\,dx}{\sqrt{1-x^2}}:$$

but $\cos u = \sqrt{1-(\sin u)^2} = \sqrt{1-4x^2+4x^4} = 1-2x^2$,

$$\text{whence we have } du = \frac{2dx}{\sqrt{1-x^2}}.$$

Ex. 5. If u [illegible] $n^{-1}\left\{\frac{1+2x}{\sqrt{3}}\right\}$, we shall manifestly

$$\text{have } \tan\frac{u\sqrt{3}}{2} = \frac{1+2x}{\sqrt{3}}:$$

$$\therefore \left(\sec\frac{u\sqrt{3}}{2}\right)^2 \frac{\sqrt{3}\,du}{2} = \frac{2dx}{\sqrt{3}}, \text{ and } du = \frac{4}{3}\,\frac{dx}{\left(\sec\frac{u\sqrt{3}}{2}\right)^2}$$

$$= \frac{4}{3}\,\frac{dx}{1+\left(\frac{1+2x}{\sqrt{3}}\right)^2} = \frac{dx}{1+x+x^2}.$$

Ex. 6. If $u = (2x^2-1)\sin^{-1}x + x\sqrt{1-x^2}$, then by (24)

$$du = 4x\sin^{-1}x\,dx + \frac{(2x^2-1)\,dx}{\sqrt{1-x^2}} + \sqrt{1-x^2}\,dx - \frac{x^2 dx}{\sqrt{1-x^2}}$$

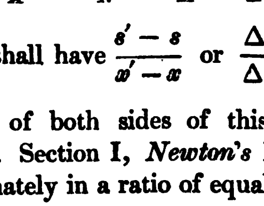
$$= 4x \sin^{-1} x\,dx + \frac{(2x^2-1)\,dx + (1-2x^2)\,dx}{\sqrt{1-x^2}} = 4x \sin^{-1} x\,dx.$$

55. The differential of each of the direct and inverse trigonometrical functions having now been investigated, we shall conclude the present Chapter by finding the differentials of the arcs, areas, &c. of any curves whatever referred to rectangular and polar co-ordinates, the general equation to the former being $y = f(x)$, or $f(x, y) = 0$, and that of the latter, $r = f(\theta)$, or $f(\theta, r) = 0$.

56. *If* s *be the Arc of a Curve referred to the rectangular co-ordinates* x *and* y, *then will*

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}, \text{ and } ds = \sqrt{dx^2 + dy^2}.$$

For, let AN and NP be the co-ordinates x and y respectively, and AP the arc s; then if $AM = x'$, $MQ = y'$,



and $AQ = s'$, we shall have $\dfrac{s'-s}{x'-x}$ or $\dfrac{\Delta s}{\Delta x} = \dfrac{\text{arc } PQ}{NM}$; whence

taking the limits of both sides of this equation, observing that by Lemma 7. Section I, *Newton's* Principia, an arc and its chord are ultimately in a ratio of equality, we obtain

$$\frac{ds}{dx} = \text{limit of } \frac{\text{arc } PQ}{NM} = \text{limit of } \frac{\text{chd } PQ}{NM}$$

$$= \text{limit of } \sqrt{\frac{PR^2 + RQ^2}{PR^2}} = \text{limit of } \sqrt{1 + \left(\frac{y'-y}{x'-x}\right)^2}$$

$$= \text{limit of } \sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2};$$

$$\text{and } \therefore\ ds = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{dx^2+dy^2}.$$

57. *If* S *be the Area of a Curve referred to the rectangular co-ordinates* x *and* y, *then will* $dS = y\,dx$.

For, retaining the figure of the last article, and using a similar notation, we have

$$\frac{S'-S}{x'-x} \text{ or } \frac{\Delta S}{\Delta x} = \frac{\text{area } NPQM}{NM};$$

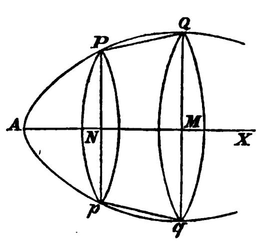
therefore taking the limits as before, we get $\frac{dS}{dx}$ = limit of

$$\frac{\text{area } NPQM}{NM} = \text{limit of } \frac{\text{trapezium } NPQM}{NM} = \text{limit of}$$

$$\frac{(NP+QM)\,NM}{2\,NM} = \text{limit of } \left(\frac{y'+y}{2}\right) = y, \text{ and } \therefore\ dS = y\,dx.$$

58. *If* Σ *be the Surface of the Solid generated by the revolution of a curve whose co-ordinates are* x *and* y *about the axis of* x, *then will* $d\Sigma = 2\pi y\,dx\sqrt{1+\left(\frac{dy}{dx}\right)^2} = 2\pi y\,ds$.

For, if the plane figure *AMQ* revolve about the axis *AX*,



we shall obviously have as before

$$\frac{\Sigma' - \Sigma}{x' - x} \text{ or } \frac{\Delta \Sigma}{\Delta x} = \frac{\text{surface generated by arc } PQ}{NM},$$

of which the limits being taken, we get

$$\frac{d\Sigma}{dx} = \text{limit of } \frac{\text{surface generated by chord } PQ}{NM}$$

$$= \text{limit of } \frac{\text{surface of frustum of cone by } PQ}{NM}$$

$$= \text{limit of } \frac{\pi (NP + MQ)\, PQ}{NM},$$ by (17) of the introductory chapter,

$$= \text{limit of } \pi (y' + y) \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}, \text{ by (56)}$$

$$= 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}; \text{ and } \therefore d\Sigma = 2\pi y dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= 2\pi y ds, \text{ by (56).}$$

59. *If* V *be the Volume or Content of the Solid generated by the revolution of a curve about the axis of* x, *then will* $dV = \pi y^2 dx$.

For, reasoning as in the preceding articles, we have

$$\frac{V' - V}{x' - x} \text{ or } \frac{\Delta V}{\Delta x} = \frac{\text{solid generated by } NPQM}{NM},$$

and the limits being taken, we obtain

$$\frac{dV}{dx} = \text{limit of } \frac{\text{solid generated by } NPQM}{NM}$$

$$= \text{limit of } \frac{\text{frustum of cone by } NPQM}{NM}$$

$$= \text{limit of } \frac{\pi NM(NP^2+NP.MQ+MQ^2)}{3NM}, \text{ by (21) of the}$$

introductory chapter,

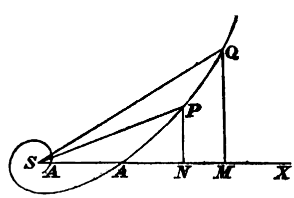
$$= \text{limit of } \pi\left(\frac{y^2+yy'+y'^2}{3}\right) = \pi y^2, \text{ and } \therefore\ dV = \pi y^2 dx.$$

60. Cor. If V represent the content of any solid, which is not generated by the revolution of a plane area about an axis, then if $\phi(y)$ denote the area of a section of the solid made by a plane perpendicular to the axis of x, it is manifest from the reasoning just employed that we need only substitute $\phi(y)$ in the place of πy^2 in the last article, and therefore $dV = \phi(y)\,dx$.

61. *If* s *be the Arc of a Curve referred to the polar coordinates* θ *and* r, *then will*

$$ds = d\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} = \sqrt{r^2 d\theta^2 + dr^2}.$$

For, if S be the pole of the curve AP, $\angle PSX = \theta$,



$SP = r$ and arc $AP = s$: then if we suppose $QSX = \theta'$, $SQ = r'$ and arc $AQ = s'$, we shall have

$$\frac{s'-s}{\theta'-\theta} \text{ or } \frac{\Delta s}{\Delta\theta} = \frac{\text{arc } PQ}{\Delta\theta}:$$

and taking the limits of both sides, we obtain

$$\frac{ds}{d\theta} = \text{limit of } \frac{\text{arc } PQ}{\Delta\theta} = \text{limit of } \frac{\text{chord } PQ}{\Delta\theta}$$

$$= \text{limit of } \frac{\sqrt{SP^2 + SQ^2 - 2\,SP.SQ\cos\Delta\theta}}{\Delta\theta}$$

$$= \text{limit of } \frac{\sqrt{r'^2 + r^2 - 2rr'\cos\Delta\theta}}{\Delta\theta}$$

$$= \text{limit of } \frac{\sqrt{(r'-r)^2 + 2rr'(1-\cos\Delta\theta)}}{\Delta\theta}$$

$$= \text{limit of } \sqrt{\left(\frac{\Delta r}{\Delta\theta}\right)^2 + rr'\left(\frac{\sin\frac{1}{2}\Delta\theta}{\frac{1}{2}\Delta\theta}\right)^2}$$

$$= \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}, \text{ and } \therefore ds = d\theta\sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}$$

$$= \sqrt{dr^2 + r^2 d\theta^2}.$$

62. *If* S *be the Area of a Curve referred to the polar co-ordinates* θ *and* r, *then will* $dS = \frac{1}{2}r^2 d\theta$.

For, retaining the figure and notation of the last article, we have

$$\frac{S'-S}{\theta'-\theta} \text{ or } \frac{\Delta S}{\Delta\theta} = \frac{\text{area } PSQ}{\Delta\theta}:$$

and taking the limits as before, we get

$$\frac{dS}{d\theta} = \text{limit of } \frac{\text{area } PSQ}{\Delta\theta} = \text{limit of } \frac{\text{triangle } PSQ}{\Delta\theta}$$

$$= \text{limit of } \frac{SP.SQ\sin\Delta\theta}{2\Delta\theta} = \text{limit of } \frac{rr'\sin\Delta\theta}{2\Delta\theta}$$

$$= \frac{r^2}{2}, \text{ by (2) of the introductory chapter,}$$

$$\text{and } \therefore dS = \tfrac{1}{2}r^2 d\theta.$$

63. In the latter articles of this chapter, the areas of plane figures, and the curved surfaces and contents of solids have been compared with lines, but it may be observed that the

methods of proof would not have been different had all the quantities employed been rendered homogeneous by the introduction of constant multipliers or divisors: and moreover, if their numerical values be understood, objections of this nature will cease to have any weight.

64. From the relation of the co-ordinates which is expresssed by the equation to the curve, all the expressions just investigated may readily be exhibited in terms of the principal variable and constant magnitudes: thus, in the case of a circle, when the co-ordinates are measured from the centre, we have $y = \sqrt{a^2 - x^2}$:

$$\therefore \frac{dy}{dx} = -\frac{x}{\sqrt{a^2-x^2}}, \text{ and } 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{a^2-x^2} = \frac{a^2}{a^2-x^2}:$$

$$\therefore ds = dx\sqrt{\frac{a^2}{a^2-x^2}} = \frac{a\,dx}{\sqrt{a^2-x^2}}:$$

$$dS = y\,dx = \sqrt{a^2-x^2}\,dx:$$

$$d\Sigma = 2\pi y\,ds = 2\pi\sqrt{a^2-x^2}\,\frac{a\,dx}{\sqrt{a^2-x^2}} = 2\pi a\,dx:$$

and $dV = \pi y^2 dx = \pi(a^2 - x^2)\,dx$.

Again, in the spiral of Archimedes, we have $r = a\theta$, and $\therefore \dfrac{dr}{d\theta} = a$, whence

$$ds = d\theta\sqrt{r^2 + a^2} = a\,d\theta\sqrt{1+\theta^2}: \text{ and } dS = \tfrac{1}{2}a^2\theta^2 d\theta.$$

CHAP. V.

On successive Differentiations. On the Elimination of constant Quantities and irrational or transcendental Functions. On the Developement of Functions.

I. SUCCESSIVE DIFFERENTIATIONS.

65. THE differential of u which is considered as representing any function of x having been determined in all kinds of quantities, it must have been observed, that the result which is the limit of $\frac{u'-u}{h}$, and which was called the differential coefficient, is in all cases either a new function of x, or a constant quantity: this function may therefore be made to undergo the same kind of operation as was performed upon the original, and consequently the limit of the ratio of its increment to that of the principal variable, or the differential coefficient of this differential coefficient may be obtained by means of the formulæ already given: the same may be said of this last differential coefficient, and it is manifest that we may continue to repeat the operation as long as the function resulting from differentiation continues to involve the principal variable x.

If then we suppose the successive differential coefficients so obtained to be represented by p, q, r, &c. we shall, according to the rules before laid down, have

$$\frac{du}{dx} = p, \quad d\left(\frac{du}{dx}\right) = dp, \quad \frac{d\left(\frac{du}{dx}\right)}{dx} = \frac{dp}{dx} = q,$$

$$d\left\{\frac{d\left(\frac{du}{dx}\right)}{dx}\right\} = dq, \quad \frac{d\left\{\frac{d\left(\frac{du}{dx}\right)}{dx}\right\}}{dx} = \frac{dq}{dx} = r, \text{ and so on:}$$

but as this notation would manifestly at length become exceedingly inconvenient, it is usual to assign to dx a constant but indeterminate value, such that the relative values of du and dx may still remain the same, and then we have

$$d\left(\frac{du}{dx}\right) = \frac{d\,(du)}{dx} = \frac{ddu}{dx}, \text{ which is written } \frac{d^2u}{dx};$$

$$\text{and } \therefore \frac{d\left(\frac{du}{dx}\right)}{dx} = \frac{d^2u}{dx\,dx} = \frac{d^2u}{dx^2}:$$

Again,

$$d\left\{\frac{d\left(\frac{du}{dx}\right)}{dx}\right\} = \frac{d\left\{d\left(\frac{du}{dx}\right)\right\}}{dx} = \frac{d\,(d^2u)}{dx^2} = \frac{d^3u}{dx^2}, \text{ as before,}$$

$$\text{and } \therefore \frac{d\left\{\frac{d\left(\frac{du}{dx}\right)}{dx}\right\}}{dx} = \frac{d^3u}{dx^3}, \text{ and so on:}$$

where it must be observed that the indices of the letter d express merely the repetition of the operation of differentiation so many times, and not powers of d, which is in this subject never used otherwise than to indicate a particular operation upon the quantity to which it is prefixed.

From the original function therefore, we have derived a series of quantities called successive differential coefficients represented by the expressions $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, $\frac{d^3u}{dx^3}$, &c. $\frac{d^nu}{dx^n}$, such that $\frac{du}{dx}=p$, $\frac{d^2u}{dx^2}=q$, $\frac{d^3u}{dx^3}=r$, &c. p, q, r, &c. being all functions of x.

66. The quantities represented by

$$\frac{du}{dx},\ \frac{d^2u}{dx^2},\ \frac{d^3u}{dx^3},\ \text{\&c. } \frac{d^nu}{dx^n},$$

are called the first, second, third, &c. n^{th} differential coefficients of the function u; and they are usually read, du by dx, second du by $(dx)^2$, third du by $(dx)^3$, &c. n^{th} du by $(dx)^n$. Moreover d^2u, d^3u, &c. d^nu must always be carefully distinguished from du^2, du^3, &c. du^n which denote the powers of du, and also from $d(u^2)$, $d(u^3)$, &c. $d(u^n)$ which express the differentials of the powers of u.

Ex. 1. Let $u = ax^3 - bx^2 + cx - e$, then the first differential coefficient $\frac{du}{dx} = 3ax^2 - 2bx + c$, which is evidently a new function of x: again, the second differential coefficient $\frac{d^2u}{dx^2} = 6ax - 2b$, which is also a new function of x; and the third differential coefficient $\frac{d^3u}{dx^3} = 6a$, which is an invariable quantity: and $\therefore$ all the succeeding differential coefficients represented by $\frac{d^4u}{dx^4}$, $\frac{d^5u}{dx^5}$, &c. $\frac{d^nu}{dx^n}$, vanish.

Whence, it appears, that this function has three differential coefficients and no more, represented by $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$ and $\frac{d^3u}{dx^3}$, whose values are $3ax^2-2bx+c$, $6ax-2b$ and $6a$.

Ex. 2. If we have $u = ax^n$ for the original function, by a repetition of the operation of differentiation, we get

$$\frac{du}{dx} = nax^{n-1}, \quad \frac{d^2u}{dx^2} = n(n-1)\,ax^{n-2},$$

$$\frac{d^3u}{dx^3} = n(n-1)(n-2)\,ax^{n-3}, \text{ \&c.}$$

$$\frac{d^nu}{dx^n} = n(n-1)(n-2) \text{ \&c. } 3\,.\,2\,.\,1\,.\,a;$$

and these are all functions of x except the last which is constant, and consequently all the succeeding differential coefficients become $= 0$.

Ex. 3. If $u = a^x$, then will $\dfrac{du}{dx} = ka^x$, $\dfrac{d^2u}{dx^2} = k^2a^x$,

$$\frac{d^3u}{dx^3} = k^3a^x, \text{ \&c. } \frac{d^nu}{dx^n} = k^na^x:$$

also if a become e, k becomes 1, and each of the differential coefficients is then equal to the original function e^x.

Ex. 4. If $u = \log x$, we shall have

$$\frac{du}{dx} = \frac{1}{x}, \quad \frac{d^2u}{dx^2} = -\frac{1}{x^2}, \quad \frac{d^3u}{dx^3} = \frac{1\,.\,2}{x^3}, \text{ \&c.}$$

and by the same process continued, we obtain

$$\frac{d^nu}{dx^n} = \pm\frac{1\,.\,2\,.\,3\,.\,\text{\&c.}\,(n-1)}{x^n},$$

wherein the positive or negative sign is to be used according as n is odd or even, and which may therefore be written generally

$$-\frac{1\,.\,2\,.\,3 \text{ \&c. } (n-1)}{x^n}(-1)^n.$$

Ex. 5. Let us take the implicit function $u^2 - 2mux + x^2 - a^2 = 0$, then will $2udu - 2mxdu - 2mudx + 2xdx = 0$,

$$\text{and } \therefore (u-mx)\frac{du}{dx}-(mu-x)=0,$$

$$\text{whence } \frac{du}{dx}=\frac{mu-x}{u-mx}=-\frac{mu-x}{mx-u}:$$

again, from $(u-mx)\dfrac{du}{dx}-(mu-x)=0$, we obtain

$$(du-mdx)\frac{du}{dx}+(u-mx)\frac{d^2u}{dx}-(mdu-dx)=0,$$

$$\text{and } \therefore \left(\frac{du}{dx}-m\right)\frac{du}{dx}+(u-mx)\frac{d^2u}{dx^2}-m\frac{du}{dx}+1=0,$$

$$\text{or } (u-mx)\frac{d^2u}{dx^2}+\frac{du^2}{dx^2}-2m\frac{du}{dx}+1=0:$$

whence by substitution and reduction, we find

$$\frac{d^2u}{dx^2}=-(m^2-1)\frac{u^2-2mux+x^2}{(mx-u)^3}:$$

and similarly of succeeding differential coefficients.

Ex. 6. To find the successive differentials of the product of two functions such as $u=pq$, we have

$$du=qdp+pdq=pq\left(\frac{dp}{p}+\frac{dq}{q}\right);$$

$$\begin{aligned}\therefore\ d^2u &= qd^2p+dpdq+dpdq+pd^2q\\ &= qd^2p+2dpdq+pd^2q\\ &= pq\left\{\frac{d^2p}{p}+2\frac{dp}{p}\frac{dq}{q}+\frac{d^2q}{q}\right\}: \text{ and so on,}\end{aligned}$$

and if we suppose generally that

$$d^nu=pq\left\{\frac{d^np}{p}+n\frac{d^{n-1}p}{p}\frac{dq}{q}+\frac{n(n-1)}{1.2}\frac{d^{n-2}p}{p}\frac{d^2q}{q}+\&c.\right.$$

$$\left.+n\frac{dp}{p}\frac{d^{n-1}q}{q}+\frac{d^nq}{q}\right\}$$

$$= qd^n p + ndqd^{n-1}p + \&c. + ndpd^{n-1}q + pd^n q,$$

we shall immediately obtain

$$d^{n+1}u = qd^{n+1}p + nd^n pdq + \frac{n(n-1)}{1.2} d^{n-1}pd^2q + \&c.$$

$$+ d^n pdq + nd^{n-1}pd^2q + \&c.$$

$$= qd^{n+1}p + (n+1)d^n pdq + \frac{(n+1)n}{1.2} d^{n-1}pd^2q + \&c.$$

$$= pq\left\{\frac{d^{n+1}p}{p} + (n+1)\frac{d^n p}{p}\frac{dq}{q} + \frac{(n+1)n}{1.2}\frac{d^{n-1}p}{p}\frac{d^2q}{q} + \&c. \right.$$

$$\left. + (n+1)\frac{dp}{p}\frac{d^n q}{q} + \frac{d^{n+1}q}{q}\right\},$$

which shews that if the form be true for any one value of n, it will be true for the next succeeding value: but it is true when $n = 1$ or 2, and therefore it is generally true.

The same mode of proceeding may manifestly be pursued in other instances; but more examples are now unnecessary as the differential coefficients of a variety of functions will be found in the applications of the Theorems hereafter to be investigated.

II. ELIMINATION OF CONSTANT QUANTITIES AND IRRATIONAL OR TRANSCENDENTAL FUNCTIONS.

67. We have already seen that every equation between two variables u and x expressed generally by $u = f(x)$, or $f(x, u) = 0$, in consequence of the operation of differentiation, gives rise to new relations between u, x and the differential coefficients $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, &c. co-existent with that expressed by the proposed equation. As therefore the constants involved in the primitive function remain the same through those successively derived from it, we shall obviously be able by means of these new equations always to eliminate from the primitive

a number of constant magnitudes expressed by the units in the number which denotes the highest order of differentiation performed. Thus, if the function proposed be $f(x, u, a, b, \&c.\ l) = 0$, in which n constant quantities are involved, there may readily be derived from it by differentiation n equations of the following forms:

$$(1).\quad f_1\left(x, u, \frac{du}{dx}, a, b, \&c.\ l\right) = 0:$$

$$(2).\quad f_2\left(x, u, \frac{du}{dx}, \frac{d^2u}{dx^2}, a, b, \&c.\ l\right) = 0:$$

$$(3).\quad f_3\left(x, u, \frac{du}{dx}, \frac{d^2u}{dx^2}, \frac{d^3u}{dx^3}, a, b, \&c.\ l\right) = 0:$$

$$\&c.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0:$$

$$(n).\quad f_n\left(x, u, \frac{du}{dx}, \frac{d^2u}{dx^2}, \frac{d^3u}{dx^3}, \&c.\ \frac{d^nu}{dx^n}, a, b, \&c.\ l\right) = 0:$$

each of which comprises the n constant quantities contained in the original equation.

Now it is obvious that by means of the proposed equation and (1) any one of the constants involved may be eliminated, and the elimination of each constant will produce a differential equation which is called a *Derivative* of the given or primitive function: and hence it follows that there may result n derivatives involving only the first differentials, which on that account are styled *Derivatives of the First Order.*

Again, by means of the primitive equation, and (1) and (2), any two of the constants concerned may be exterminated, and thus give rise to an equation involving first and second differentials, or what is called a *Derivative of the Second Order:* and it is manifest that the number of derivatives of this order will be equal to that of the combinations which

can be formed out of n things taken two together, or $\frac{n(n-1)}{1.2}$, as appears from *Algebra* (259).

Similarly, the number of derivatives of the third order will be expressed by $\frac{n(n-1)(n-2)}{1.2.3}$, and so on: and if the whole of the n constants be eliminated, the number of derivatives of the n^{th} order will manifestly be

$$\frac{n(n-1)(n-2).\&c.\,3.2.1}{1.2.3.\&c.\,(n-2)(n-1)n}, \text{ or } 1.$$

We will now illustrate these principles by a few examples.

Ex. 1. Let us take the simple equation $u-ax+b=0$, which involves the constants a and b: then $du-adx=0$, which is a derivative of the first order not involving b: whence $a=\frac{du}{dx}$, and this by substitution in the primitive equation gives $u-\frac{xdu}{dx}+b=0$, or $udx-xdu+bdx=0$, which is also a derivative of the first order independent of a: and thus have we obtained in this case two derivative equations of the first order.

Again, $u-ax+b=0$, $du-adx=0$, and $d^2u=0$, which last is a derivative of the second order, and from this both a and b have disappeared.

Ex. 2. Let the equation proposed be $u^2=m(a^2-x^2)$; then we have $udu=-mxdx$, or $udu+mxdx=0$, which is a derivative of the first order not involving a: and since $m=-\frac{udu}{xdx}$, if this value be substituted in the proposed equation, we get

$$u^2 = -\frac{udu}{xdx}(a^2 - x^2), \text{ or } u^2 xdx + (a^2 - x^2)\, udu = 0,$$

which is another derivative of the first order independent of m.

Differentiating again, we find $ud^2u + du^2 = -mdx^2$; and substituting $-\dfrac{udu}{xdx}$ in the place of m, we arrive at the equation

$$u\frac{du}{dx} - x\frac{du^2}{dx^2} - xu\frac{d^2u}{dx^2} = 0, \text{ or } ududx - xdu^2 - xud^2u = 0,$$

which is a derivative of the second order independent of both a and m.

Ex. 3. If the proposed equation be $(x-a)^2 + (y-b)^2 - c^2 = 0$, we find the derived equation of the first order to be $x - a + (y-b)\dfrac{dy}{dx} = 0$, which does not contain c; and it is obvious that by means of it and the original, there may be obtained three derivatives of the first order, from which a, b and c will be respectively eliminated.

Again, the derived equations of the second and third orders are

$$1 + (y-b)\frac{d^2y}{dx^2} + \frac{dy^2}{dx^2} = 0, \text{ and } (y-b)\frac{d^3y}{dx^3} + 3\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} = 0,$$

which with those above readily lead to three derivatives of the second order, and one of the third.

68. Cor. Precisely in the same manner there may be made to disappear any number of irrational and transcendental functions, by combining the proposed equations with those derived from them by differentiation.

Ex. 1. Let $u = (x + \sqrt{x^2 - 1})^m$, then will

$$\log u = m \log (x + \sqrt{x^2 - 1}), \text{ and } \frac{du}{u} = \frac{m\,dx}{\sqrt{x^2 - 1}}:$$

whence $(x^2 - 1)\,du^2 = m^2 u^2 dx^2$, which is free from surds.

Ex. 2. If $u = \dfrac{e^{2x} + 1}{e^{2x} - 1}$, we shall have

$$e^{2x} = \frac{u+1}{u-1}, \text{ and } 2x = \log\left(\frac{u+1}{u-1}\right):$$

$$\text{whence } dx = \frac{du}{1-u^2}, \text{ and } (1-u^2)\,dx - du = 0,$$

which involves no exponentials.

Ex. 3. Let $u = a \sin x + b \cos x$, then will

$$\frac{du}{dx} = -a \cos x + b \sin x, \quad \frac{d^2u}{dx^2} = -a \sin x + b \cos x = -u;$$

and $\therefore$ $d^2u + u\,dx^2 = 0$, from which both the constants and transcendentals have been made to disappear.

III. Developement of Functions, &c.

69. *If* u *which represents a function of* x *can be expressed in ascending integral and positive powers of* x, *its developement may be obtained from the equation*

$$\mathrm{u} = \mathrm{U}_0 + \mathrm{U}_1 \frac{\mathrm{x}}{1} + \mathrm{U}_2 \frac{\mathrm{x}^2}{1.2} + \mathrm{U}_3 \frac{\mathrm{x}^3}{1.2.3} + \&\text{c.} + \mathrm{U}_n \frac{\mathrm{x}^n}{1.2.3.\&\text{c. } n},$$

where $\mathrm{U}_0, \mathrm{U}_1, \mathrm{U}_2, \mathrm{U}_3$, &c. U_n *are the particular values of the function* u, *and its differential coefficients*

$$\frac{d\mathrm{u}}{d\mathrm{x}}, \frac{d^2\mathrm{u}}{d\mathrm{x}^2}, \frac{d^3\mathrm{u}}{d\mathrm{x}^3}, \&\text{c.}, \frac{d^n\mathrm{u}}{d\mathrm{x}^n}, \text{ when x is made} = 0.$$

For, let $u = A + Bx + Cx^2 + Dx^3 + \&c.$, where A, B, C, D, &c. are quantities which do not involve x; then

$$\frac{du}{dx} = 1\,B + 2\,Cx + 3\,Dx^2 + 4\,Ex^3 + \&c.$$

$$\frac{d^2u}{dx^2} = 1.2\,C + 2.3\,Dx + 3.4\,Ex^2 + \&c.$$

$$\frac{d^3u}{dx^3} = 1.2.3\,D + 2.3.4\,Ex + \&c.$$

$$\frac{d^4u}{dx^4} = 1.2.3.4\,E + \&c.$$

$$\&c. = \&c.$$

Now, since A, B, C, D, &c. do not involve x, they must remain the same, whatever value be assigned to x, and therefore when x is made $= 0$; but on this hypothesis, we have

$$U_0 = A, \text{ or } A = U_0; \quad U_1 = 1\,B, \text{ or } B = \frac{U_1}{1};$$

$$U_2 = 1.2\,C, \text{ or } C = \frac{U_2}{1.2}; \quad U_3 = 1.2.3\,D, \text{ or } D = \frac{U_3}{1.2.3};$$

$$\&c. = \&c. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \&c. = \&c.$$

$$\text{whence } u = U_0 + U_1\frac{x}{1} + U_2\frac{x^2}{1.2} + U_3\frac{x^3}{1.2.3} + \&c.$$

$+ U_n \dfrac{x^n}{1.2.3.\&c.\,n} + \&c.$, which is known by the name of *Maclaurin's* Theorem.

Sometimes the particular values of u and its differential coefficients are expressed by means of brackets, and the theorem is thus written

$$u = (u) + \left(\frac{du}{dx}\right)\frac{x}{1} + \left(\frac{d^2u}{dx^2}\right)\frac{x^2}{1.2} + \left(\frac{d^3u}{dx^3}\right)\frac{x^3}{1.2.3} + \&c.$$

Ex. 1. Let $u = (a+x)^m$, therefore $U_0 = a^m$;

$$\text{also, } \frac{du}{dx} = m(a+x)^{m-1}, \ \therefore \ U_1 = ma^{m-1};$$

$$\frac{d^2u}{dx^2} = m(m-1)(a+x)^{m-2}, \ \therefore \ U_2 = m(m-1)\,a^{m-2};$$

$$\frac{d^3u}{dx^3} = m(m-1)(m-2)(a+x)^{m-3}, \ \therefore \ U_3 = m(m-1)(m-2)a^{m-3};$$

$$\&\text{c.} = \&\text{c.} \qquad\qquad \&\text{c.} = \&\text{c.}$$

whence u or $(a+x)^m = a^m + ma^{m-1}x + \dfrac{m(m-1)}{1.2}a^{m-2}x^2$

$\dfrac{m(m-1)(m-2)}{1.2.3}a^{m-3}x^3 + \&\text{c.}$, which is the binomial theorem: and this is a complete proof because in (18) the differential of x^m has been independently determined.

Ex. 2. Let u be any rational function whatever of x in which the highest power of x is m, then it is obvious that we may assume $u = f(x-a)$, wherein a is indeterminate; hence we shall have

$$u = U_0 + U_1\frac{(x-a)}{1} + U_2\frac{(x-a)^2}{1.2} + \&\text{c.} + U_m\frac{(x-a)^m}{1.2.3.\&\text{c.}m},$$

since the differential coefficients after the m^{th} vanish:

$$\therefore \ \frac{u}{(x-a)^m} = \frac{U_0}{(x-a)^m} + \frac{U_1}{1\,(x-a)^{m-1}} + \frac{U_2}{1.2\,(x-a)^{m-2}} + \&\text{c.}$$

$+\dfrac{U_m}{1.2.3.\&\text{c.}m}$, where U_0, U_1, U_2, &c. U_m are the values of the proposed function and its successive differential coefficients when $x-a=0$, or $x=a$.

Ex. 3. Let $u = a^x$, therefore $U_0 = 1$;

also, $\dfrac{du}{dx} = ka^x$, $\therefore$ $U_1 = k$; $\dfrac{d^2u}{dx^2} = k^2a^x$, $\therefore$ $U_2 = k^2$;

$\dfrac{d^3u}{dx^3} = k^3a^x$, $\therefore$ $U_3 = k^3$; &c. = &c.

$$\therefore u \text{ or } a^x = 1 + \frac{kx}{1} + \frac{k^2x^2}{1.2} + \frac{k^3x^3}{1.2.3} + \text{\&c.} + \frac{k^nx^n}{1.2.3.\text{\&c.}n} + \text{\&c.}$$

Ex. 4. Let $u = \log(a+x)$, $\therefore$ $U_0 = \log a$;

$$\text{also, } \frac{du}{dx} = \frac{1}{a+x} = \frac{1}{a} - \frac{x}{a^2} + \frac{x^2}{a^3} - \frac{x^3}{a^4} + \frac{x^4}{a^5} - \text{\&c.};$$

$$\therefore \frac{d^2u}{dx^2} = -\frac{1}{a^2} + 2\frac{x}{a^3} - 3\frac{x^2}{a^4} + 4\frac{x^3}{a^5} - \text{\&c.}:$$

$$\frac{d^3u}{dx^3} = 1.2\frac{1}{a^3} - 2.3\frac{x}{a^4} + 3.4\frac{x^2}{a^5} - \text{\&c.}:$$

$$\frac{d^4u}{dx^4} = -1.2.3\frac{1}{a^4} + 2.3.4\frac{x}{a^5} - \text{\&c.}$$

&c. = &c.

whence $U_1 = \dfrac{1}{a}$, $U_2 = -\dfrac{1}{a^2}$, $U_3 = 1.2\dfrac{1}{a^3}$, $U_4 = -1.2.3\dfrac{1}{a^4}$, &c.

$$\text{and } \therefore \log(a+x) = \log a + \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \text{\&c.}$$

If $a=1$, we obtain $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \text{\&c.}$

Ex. 5. Let $u = \sin(x+a)$, $\therefore$ $U_0 = \sin a$;

also $\dfrac{du}{dx} = \cos(x+a)$, $\dfrac{d^2u}{dx^2} = -\sin(x+a)$, $\dfrac{d^3u}{dx^3} = -\cos(x+a)$, &c.

whence we have $U_1 = \cos a$, $U_2 = -\sin a$, $U_3 = -\cos a$, &c. and therefore u or $\sin(x+a)$

$$= \sin a + \cos a\,\frac{x}{1} - \sin a\,\frac{x^2}{1.2} - \cos a\,\frac{x^3}{1.2.3} + \text{\&c.}:$$

and if $a = 0$, then $\sin a = 0$ and $\cos a = 1$;

$$\text{whence } \sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \text{\&c.}$$

Similarly of $\cos(a+x)$ and $\cos x$.

Ex. 6. If $u = \sin^{-1}x$, we shall obviously have

$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}, \text{ by (51)}$$

$$=1+\tfrac{1}{2}x^2+\frac{1.3}{2.4}x^4+\frac{1.3.5}{2.4.6}x^6+\text{\&c.}+\frac{1.3.5.\text{\&c.}(2n-3)}{2.4.6.\text{\&c.}(2n-2)}x^{2n-2}+\text{\&c.}$$

by the binomial theorem,

$$= 1 + a_2x^2 + a_4x^4 + a_6x^6 + \text{\&c.} + a_{2n-2}x^{2n-2} + \text{\&c. suppose}:$$

$$\therefore\ \frac{d^2u}{dx^2} = 2a_2x + 4a_4x^3 + 6a_6x^5 + \text{\&c.} + (2n-2)\,a_{2n-2}x^{2n-3} + \text{\&c.}$$

$$\frac{d^3u}{dx^3} = 1.2a_2 + 3.4a_4x^2 + 5.6a_6x^4 + \text{\&c.} + (2n-3)(2n-2)\,a_{2n-2}x^{2n-4}$$

$+$ &c., and so on,

$$\frac{d^{2n-1}u}{dx^{2n-1}} = 1.2.3.\text{\&c.}\ (2n-3)\ (2n-2)\ a_{2n-2} + \text{\&c.}:$$

whence if x be assumed $=0$, and the values of a_2, a_4, &c., a_{2n-2}, be restored, we shall have

$$U_0 = 0, \quad U_1 = 1, \quad U_2 = 0, \; U_3 = 1, \; \text{\&c.} = \text{\&c.},$$

$$U_{2n-1} = 1.2.3.\text{\&c.}\,(2n-3)\,(2n-2) \times \frac{1.3.5.\text{\&c.}\,(2n-3)}{2.4.6.\text{\&c.}\,(2n-2)},$$

$$\therefore u = x + \frac{x^3}{2.3} + \text{\&c.} + \frac{1.3.5.\text{\&c.}\,(2n-3)\,x^{2n-1}}{2.4.6.\text{\&c.}\,(2n-2)\,(2n-1)} + \text{\&c.}$$

which comprehends the general term of the developement.

Ex. 7. Let u be an implicit function of x, such that $u^2 - 2ux - a^2 = 0$, then will

$$\frac{du}{dx} = \frac{u}{u-x}, \; \frac{d^2u}{dx^2} = \frac{u^2 - 2ux}{(u-x)^3}, \; \frac{d^3u}{dx^3} = -\frac{3ux\,(u-2x)}{(u-x)^5}, \; \text{\&c.}$$

and if $x = 0$, we shall have corresponding thereto, $U_0 = \pm a$,

and therefore $U_1 = 1$, $U_2 = \pm \frac{1}{a}$, $U_3 = 0$, &c. = &c.

whence $u = U_0 + U_1 \frac{x}{1} + U_2 \frac{x^2}{1.2} + U_3 \frac{x^3}{1.2.3} + \text{\&c.}$

$$= \pm a + x \pm \frac{x^2}{2a} \mp \text{\&c.}$$

and this expresses in series the two roots of the quadratic, as might have been found by solving the equation with respect to u, and then expanding the result into a series.

Ex. 8. If $u^3 - ux^2 - a^3 = 0$, be the proposed implicit function, and the differential coefficients be found as before, we shall on the hypothesis of x being made $= 0$, have $U^3_0 - a^3 = 0$, whence U_0 will have the three values a, αa, βa where α and β are $-\frac{1+\sqrt{-3}}{2}$ and $-\frac{1-\sqrt{-3}}{2}$ respectively; and thus u will be developed in three different series which are the roots of the proposed equation.

If it were required to develope u in a series ascending by the powers of a, we should have to consider u as a function of a, and then on the supposition that $a=0$, should obtain

$$U^3_0 - U_0 x^2 = 0, \text{ whence } U_0 = 0 \text{ and } U_0 = \pm x;$$

and the corresponding values of U_1, U_2, U_3, &c. might be found as in the last example, which would give rise to three different series as before.

Ex. 9. Let $u = \sqrt{2x-1}$, $\therefore \dfrac{du}{dx} = \dfrac{1}{(2x-1)^{\frac{1}{2}}}$,

$$\frac{d^2u}{dx^2} = -\frac{1}{(2x-1)^{\frac{3}{2}}}, \quad \frac{d^3u}{dx^3} = \frac{1.3}{(2x-1)^{\frac{5}{2}}}, \text{ \&c.};$$

whence we have $U_0 = \sqrt{-1}$, $U_1 = -\sqrt{-1}$, $U_2 = -\sqrt{-1}$, $U_3 = -3\sqrt{-1}$, &c.

$$\text{and } \therefore \sqrt{2x-1} = \sqrt{-1}\left\{1 - x - \frac{x^2}{2} - \frac{x^3}{2} - \text{\&c.}\right\};$$

but it may be observed that this function is capable of being developed by other means in possible terms, but not in integral ascending powers of x: and indeed the impossibility of effecting the developement in the form proposed by the theorem is indicated by the symbol $\sqrt{-1}$ which appears in each of the terms.

70. Cor. 1. It has been proved in Ex. (4) that

$$\log(a+x) = \log a + \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \text{\&c.}:$$

and by supposing $a=0$, we should obviously have

$$\log x = -\infty + \infty - \infty + \text{\&c.},$$

from which expression, though it *may* be finite, nothing can be determined, and therefore in this particular instance *Maclaurin's* Theorem fails to give us the developement.

We nevertheless have

$$\log x = \log\{1+(x-1)\} = (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \text{\&c.}$$

by putting 1 and $x-1$ in the places of a and x respectively; but still if the latter member of this equation be arranged according to ascending powers of x, the coefficient of each term will be indefinitely great, so that in fact *log* x cannot be expanded at all in the form contemplated by the theorem, the coefficients being finite.

Whenever therefore the particular values U_0, U_1, U_2, &c. of the function or its differential coefficients become infinite, *Maclaurin's* Theorem is said to *fail*, and the circumstance merely points out the impossibility of developing the function in the proposed form: it is moreover easily shewn conversely that if the expansion should involve fractional or negative powers of the principal variable, the function or some of its differential coefficients must become infinite when the particular value of the principal variable used in the theorem is assigned to it: thus, if $u = ax^{\frac{5}{2}} + bx^{\frac{1}{2}} + cx^{-1} + \&c.$, we shall immediately have

$$\frac{du}{dx} = \tfrac{5}{2}\,ax^{\frac{3}{2}} + \tfrac{1}{2}\,bx^{-\frac{1}{2}} - cx^{-2} + \&c.$$

$$\frac{d^2u}{dx^2} = \tfrac{15}{4}\,ax^{\frac{1}{2}} - \tfrac{1}{4}\,bx^{-\frac{3}{2}} + 2\,cx^{-3} + \&c., \ \&c. = \&c.:$$

whence it follows that each of the quantities denoted by U_0, U_1, U_2, &c. is infinite.

71. Cor. 2. It is obvious however that if such a factor can be found as shall render the function capable of developement in ascending integral powers of the principal variable, the Theorem of *Maclaurin* may still be successfully applied: and this will manifestly be the case, whenever u can be assumed $= x^k v$, so that none of the particular values V_0, V_1, V_2, &c. of v may become infinite.

Ex. 1. Let us take the simple instance, $u = \sqrt{x - x^2}$, from which we should immediately deduce

$$U_0 = 0, \ U_1 = \infty, \ U_2 = -\infty, \ U_3 = \infty, \ \&c. = \&c.$$

$$Ba=b, \text{ and } \therefore B=\frac{b}{a}; \quad 2Ca+Bb=2c,$$

$$\text{and } \therefore C=\frac{2c-Bb}{2a}=\frac{b^2-2ac}{2a^2};$$

$$3D+2Cb+Bc=0, \text{ and } \therefore D=-\frac{2Cb+Bc}{3}=\frac{b^3-3abc}{3a^2};$$

$$4E+3Db+2Cc=0,$$

$$\text{and } \therefore E=-\frac{3Db+2Cc}{4}=-\frac{b^4-3ab^2c+2ac^2-b^2c}{4a^2}, \text{ \&c.},$$

also if $x=0$, we find $A=\log a$; whence we obtain

$$\log(a+bx+cx^2)=\log a+\frac{b}{a}x-\frac{b^2-2ac}{2a^2}x^2$$

$$+\frac{b^3-3abc}{3a^2}x^3-\frac{b^4-3ab^2c+2ac^2-b^2c}{4a^2}x^4+\text{ \&c.}$$

Similarly, by taking logarithms, the expansion of the multinomial $(a+bx+cx^2+\text{ \&c.})^m$ may be determined.

Ex. 3. Let us take again the function

$$u=(x+\sqrt{x^2-1})^m=A_0+A_1x+A_2x^2+A_3x^3+\text{ \&c.}:$$

then, we shall readily have by two successive differentiations,

$$\frac{du}{dx}=A_1+2A_2x+3A_3x^2+4A_4x^3+\text{ \&c.}$$

$$\text{and } \frac{d^2u}{dx^2}=1.2A_2+2.3A_3x+3.4A_4x^2+\text{ \&c.}:$$

but it is easily shewn from the proposed function that

$$(x^2-1)\frac{du^2}{dx^2}-m^2u=0, \text{ and } (x^2-1)\frac{d^2u}{dx^2}+x\frac{du}{dx}-m^2u=0:$$

in the latter of which if the values of u, $\frac{du}{dx}$ and $\frac{d^2u}{dx^2}$ above found be substituted, there will thence result

$$\{m^2A_0+2A_2\}+\{(m^2-1)A_1+2.3A_3\}x+\{(m^2-4)A_2+3.4A_4\}x^2$$
$$+\{(m^2-9)A_3+4.5A_5\}x^3+\{(m^2-16)A_4+5.6A_6\}x^4+\&c.=0:$$

from which are immediately obtained

$$A_2=-\frac{m^2}{2}A_0,\ A_3=-\frac{m^2-1}{2.3}A_1,\ A_4=-\frac{m^2-4}{3.4}A_2=\frac{m^2(m^2-4)}{2.3.4}A_0,$$

$$A_5=-\frac{m^2-9}{4.5}A_3=\frac{(m^2-1)(m^2-9)}{2.3.4.5}A_1,$$

$$A_6=-\frac{m^2-16}{5.6}A_4=-\frac{m^2(m^2-4)(m^2-16)}{2.3.4.5.6}A_0,\ \&c.$$

and it now remains only to determine the values of A_0 and A_1: but since $u=(x+\sqrt{x^2-1})^m=A_0+A_1x+A_2x^2+A_3x^3+\&c.$

$$\text{and } \frac{du}{dx}=\frac{mu}{\sqrt{x^2-1}}=A_1+2A_2x+3A_3x^2+4A_4x^3+\&c.$$

if we suppose $x=0$, we shall find

$$A_0=(\sqrt{-1})^m=(-1)^{\frac{m}{2}},\ \text{and } A_1=m(\sqrt{-1})^{m-1}=m(-1)^{\frac{m-1}{2}};$$

whence by substitution, we finally arrive at

$$(x+\sqrt{x^2-1})^m$$
$$=(-1)^{\frac{m}{2}}\left\{1-\frac{m^2}{1.2}x^2+\frac{m^2(m^2-2^2)}{1.2.3.4}x^4\right.$$
$$\left.-\frac{m^2(m^2-2^2)(m^2-4^2)}{1.2.3.4.5.6}x^6+\&c.\right\}$$
$$+m(-1)^{\frac{m-1}{2}}\left\{x-\frac{(m^2-1^2)}{1.2.3}x^3+\frac{(m^2-1^2)(m^2-3^2)}{1.2.3.4.5}x^5\right.$$
$$\left.-\frac{(m^2-1^2)(m^2-3^2)(m^2-5^2)}{1.2.3.4.5.6.7}x^7+\&c.\right\}.$$

Similarly by substituting in both sides of this last equation, $-m$ in the place of m, we shall have

$$(x + \sqrt{x^2 - 1})^{-m}$$

$$= (-1)^{-\frac{m}{2}} \left\{ 1 - \frac{m^2}{1.2} x^2 + \frac{m^2 (m^2 - 2^2)}{1.2.3.4} x^4 \right.$$

$$\left. - \frac{m^2 (m^2 - 2^2)(m^2 - 4^2)}{1.2.3.4.5.6} x^6 + \&c. \right\}$$

$$+ m (-1)^{\frac{-m-1}{2}} \left\{ x - \frac{(m^2 - 1^2)}{1.2.3} x^3 + \frac{(m^2 - 1^2)(m^2 - 3^2)}{1.2.3.4.5} x^5 \right.$$

$$\left. - \frac{(m^2 - 1^2)(m^2 - 3^2)(m^2 - 5^2)}{1.2.3.4.5.6.7} x^7 + \&c. \right\}.$$

If we make $x = \cos A$, we shall manifestly have

$$(\cos A + \sqrt{-1} \sin A)^m = \cos mA + \sqrt{-1} \sin mA,$$

and by equating possible and impossible quantities the developements of $\cos mA$ and $\sin mA$ will thus be obtained in series of ascending powers of $\cos A$.

Ex. 4. To develope $u = \tan x$, we have

$$u = \frac{\sin x}{\cos x} = \frac{x - \dfrac{x^3}{1.2.3} + \dfrac{x^5}{1.2.3.4.5} - \&c.}{1 - \dfrac{x^2}{1.2} + \dfrac{x^4}{1.2.3.4} - \&c.}$$

$$= a_1 x + a_3 x^3 + a_5 x^5 + \&c.\ a_{2n-1} x^{2n-1} + \&c.:$$

whence effecting the multiplications and equating coefficients, we find

$$a_1 = 1,\ a_3 = \frac{a_1}{1.2} - \frac{1}{1.2.3} = \frac{2}{1.2.3};$$

$$a_5 = \frac{a_3}{1.2} - \frac{a_1}{1.2.3.4} + \frac{1}{1.2.3.4.5} = \frac{16}{1.2.3.4.5};$$

$$a_7 = \frac{a_5}{1.2} - \frac{a_3}{1.2.3.4} + \frac{a_1}{1.2.3.4.5.6} - \frac{1}{1.2.3.4.5.6.7}$$

$$= \frac{272}{1.2.3.4.5.6.7};$$

and the general term is determined from the formula

$$a_{2n-1} = \frac{a_{2n-3}}{1.2} - \frac{a_{2n-5}}{1.2.3.4} + \frac{a_{2n-7}}{1.2.3.4.5.6} - \&c.$$

$$\pm \frac{a_1}{1.2.3.\&c.(2n-2)} \mp \frac{1}{1.2.3.\&c.(2n-1)},$$

which could not be found by the ordinary process.

Ex. 5. Let $u = \dfrac{x}{e^x - 1}$, which by substitution,

$$= \frac{x}{x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.} = \frac{1}{1 + \frac{x}{1.2} + \frac{x^2}{1.2.3} + \&c.}$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \&c. + a_{n-1} x^{n-1};$$

then by multiplication and equating coefficients we shall find

$$a_0 = 1, \quad a_1 = -\tfrac{1}{2}, \quad a_2 = \tfrac{1}{6}, \quad a_3 = 0, \ \&c.$$

$$\text{so that } u = 1 - \tfrac{1}{2} x + \tfrac{1}{6} \frac{x^2}{1.2} - \frac{1}{30} \frac{x^4}{1.2.3.4} + \&c.$$

and the law may be established generally as in the last example.

This function could not have been developed by means of *Maclaurin's* Theorem, without having recourse to principles not yet explained, since the values of U_0, U_1, U_2, &c. assume the form $\frac{0}{0}$ on the supposition that x is made $= 0$.

72. Cor. 4. If it be required to develope the function in descending, instead of ascending, powers of the principal

variable, it will frequently be sufficient to find its expansion in ascending powers of one of the constants involved in it as in Ex. 8. of (68); or generally by assuming $x = \frac{1}{z}$ and then finding the developement of the function by the formula

$$u = U_0 + U_1 \frac{z}{1} + U_2 \frac{z^2}{1.2} + U_3 \frac{z^3}{1.2.3} + \&c.:$$

thus, in the simple instance $u = \frac{x}{1-x}$, which by the common formula $= x + x^2 + x^3 + \&c.$, if we put $x = \frac{1}{z}$, then will

$u = \frac{1}{z-1} = (z-1)^{-1}$, and $\therefore\ U_0 = 1,\ U_1 = -1,\ U_2 = -2$, &c.

whence $u = -1 - z - z^2 - \&c. = -\left\{1 + \frac{1}{x} + \frac{1}{x^2} + \&c.\right\}$:

as appears also by the actual division of x by $-x+1$.

73. Cor. 5. If a_n represent the coefficient of x^n in the expansion of any function of x by *Maclaurin's* Theorem, and b_n in a similar expansion of the *Naperian* logarithm of that function, then will

$$na_n = b_1 a_{n-1} + 2b_2 a_{n-2} + 3b_3 a_{n-3} + \&c. + (n-1) b_{n-1} a_1 + n b_n a_0.$$

For, let us make the two general assumptions,

$$u = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \&c. + a_{n-1} x^{n-1} + a_n x^n + \&c.$$

$$\log u = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \&c. + b_{n-1} x^{n-1} + b_n x^n + \&c.$$

$$\therefore \frac{du}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + \&c. + (n-1) a_{n-1} x^{n-2} + n a_n x^{n-1} + \&c.$$

and

$$\frac{du}{udx} = b_1 + 2b_2 x + 3b_3 x^2 + \&c. + (n-1) b_{n-1} x^{n-2} + n b_n x^{n-1} + \&c.$$

whence we have immediately

$$a_1 + 2a_2x + 3a_3x^2 + \&c. + (n-1)a_{n-1}x^{n-2} + na_nx^{n-1} + \&c.$$
$$= \{a_0 + a_1x + a_2x^2 + a_3x^3 + \&c. + a_{n-1}x^{n-1} + a_nx^n + \&c.\}$$
$$\times \{b_1 + 2b_2x + 3b_3x^2 + \&c. + (n-1)b_{n-1}x^{n-2} + nb_nx^{n-1} + \&c.\}:$$

and by equating the coefficients of x^{n-1} in both sides of this equation, there results the formula

$$na_n = b_1a_{n-1} + 2b_2a_{n-2} + 3b_3a_{n-3} + \&c. + (n-1)b_{n-1}a_1 + nb_na_0:$$

by means of which, the relation between the coefficients of the expansion of log u may be determined, if that between the coefficients of the expansion of u be known.

74. *If* u′ *represent the same function of* x + h, *that* u *is of* x, *then if* u′ *can be developed in terms of* x *and integral positive ascending powers of* h, *the developement may be obtained from the equation*

$$u' = u + \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\frac{h^3}{1.2.3} + \&c. + \frac{d^nu}{dx^n}\frac{h^n}{1.2.3.\&c.\,n}$$
$$+ \&c.$$

Since, when h becomes 0, u' becomes u, it is obvious that we may assume $u' = u + Ph^\alpha + Qh^\beta + Rh^\gamma + \&c.$ wherein the quantities P, Q, R, &c. are functions of x not involving h; whence by differentiating with respect to x and h, we obtain successively

$$\frac{du'}{dx} = \frac{du}{dx} + \frac{dP}{dx}h^\alpha + \frac{dQ}{dx}h^\beta + \frac{dR}{dx}h^\gamma + \&c.$$

$$\text{and } \frac{du'}{dh} = \alpha Ph^{\alpha-1} + \beta Qh^{\beta-1} + \gamma Rh^{\gamma-1} + \&c.:$$

now since in $u' = f(x+h)$, the quantities x and h are involved in precisely the same manner, it is manifest that $\dfrac{du'}{dh} = \dfrac{du'}{dx}$,

and $\therefore\ \alpha P h^{\alpha-1} + \beta Q h^{\beta-1} + \gamma R h^{\gamma-1} + \&c.$

$$= \frac{du}{dx} + \frac{dP}{dx} h^{\alpha} + \frac{dQ}{dx} h^{\beta} + \frac{dR}{dx} h^{\gamma} + \&c.$$

whatever be the value of h; whence equating the indices and coefficients of corresponding terms, we shall have

$$\alpha P h^{\alpha-1} = \frac{du}{dx} = \frac{du}{dx} h^0, \quad \therefore\ \alpha - 1 = 0, \text{ or } \alpha = 1 \text{ and } P = \frac{du}{dx}:$$

$$\beta Q h^{\beta-1} = \frac{dP}{dx} h^{\alpha}, \quad \therefore \beta = \alpha + 1 = 2 \text{ and } Q = \frac{1}{2}\frac{dP}{dx} = \frac{1}{1.2}\frac{d^2u}{dx^2};$$

$$\text{so, } \gamma = \beta + 1 = 3 \text{ and } R = \frac{1}{3}\frac{dQ}{dx} = \frac{1}{1.2.3}\frac{d^3u}{dx^3}, \text{ and so on:}$$

$$\therefore u' = u + \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\frac{h^3}{1.2.3} + \&c. + \frac{d^nu}{dx^n}\frac{h^n}{1.2.3.\&c.\,n} + \&c.$$

Since $u = fx$ and $u' = f\,(x + h)$, this Theorem may be written

$$f(x+h) = fx + \frac{d(fx)}{dx}\frac{h}{1} + \frac{d^2(fx)}{dx^2}\frac{h^2}{1.2} + \frac{d^3(fx)}{dx^3}\frac{h^3}{1.2.3} + \&c.$$

Also, if p, q, r, &c. denote the successive differential coefficients, retaining the double sign of h, we shall have

$$f\,(x \pm h) = fx \pm p\frac{h}{1} + q\frac{h^2}{1.2} \pm r\frac{h^3}{1.2.3} + \&c.$$

This important Theorem was first given by Dr. Brook Taylor, and on that account is designated *Taylor's* Theorem.

Ex. 1. Let $u = x^m$ and $\therefore\ u' = (x + h)^m$,

then we have $\dfrac{du}{dx} = mx^{m-1}$; $\dfrac{d^2u}{dx^2} = m\,(m - 1)\,x^{m-2}$;

$$\frac{d^3u}{dx^3} = m\,(m-1)\,(m-2)\,x^{m-3};\ \ \&c. = \&c.$$

therefore u' or $(x+h)^m = x^m + mx^{m-1}h$

$$+\frac{m\,(m-1)}{1.2}x^{m-2}h^2 + \frac{m\,(m-1)\,(m-2)}{1.2.3}x^{m-3}h^3 + \&c.$$

This example is a general proof of the binomial theorem, though it is not unusual to assume the binomal theorem which can be established upon algebraical principles, to prove that of *Taylor*. Thus,

let $u = ax^\alpha + bx^\beta + cx^\gamma + \&c.$, and we shall obviously have

$$u' = a\,(x+h)^\alpha + b\,(x+h)^\beta + c\,(x+h)^\gamma + \&c.$$

$$= ax^\alpha + bx^\beta + cx^\gamma + \&c.$$

$$+\ (\alpha ax^{\alpha-1} + \beta bx^{\beta-1} + \gamma cx^{\gamma-1} + \&c.)\,\frac{h}{1}$$

$$+\{\alpha(\alpha-1)ax^{\alpha-2}+\beta(\beta-1)bx^{\beta-2}+\gamma(\gamma-1)cx^{\gamma-2}+\&c.\}\frac{h^2}{1.2}+\&c.$$

$$= u + \frac{du}{dx}\,\frac{h}{1} + \frac{d^2u}{dx^2}\,\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\,\frac{h^3}{1.2.3} + \&c.$$

by substituting for the successive lines of the expansion taken in order, the equivalent expressions

$$u,\ \frac{du}{dx}\,\frac{h}{1},\ \frac{d^2u}{dx^2}\,\frac{h^2}{1.2},\ \&c.$$

Ex. 2. Let $u = \log x$, or $u' = \log\,(x+h)$, then

$$\frac{du}{dx} = \frac{1}{x};\ \frac{d^2u}{dx^2} = -\frac{1}{x^2};\ \frac{d^3u}{dx^3} = \frac{1.2}{x^3};\ \frac{d^4u}{dx^4} = -\frac{2.3}{x^4};\ \&c.$$

$$\therefore\ u' \text{ or } \log\,(x+h) = \log x + \left(\frac{h}{x}\right) - \frac{1}{2}\left(\frac{h}{x}\right)^2 + \frac{1}{3}\left(\frac{h}{x}\right)^3 - \&c.$$

Ex. 3. Let $u = \sin x$, or $u' = \sin(x+h)$, then

$$\frac{du}{dx} = \cos x;\ \frac{d^2u}{dx^2} = -\sin x;\ \frac{d^3u}{dx^3} = -\cos x;\ \frac{d^4u}{dx^4} = \sin x;\ \&c.,$$

$$\therefore\ u' \text{ or } \sin(x+h) = \sin x + \cos x\,\frac{h}{1} - \sin x\,\frac{h^2}{1.2}$$

$$-\cos x\,\frac{h^3}{1.2.3} + \sin x\,\frac{h^4}{1.2.3.4} + \&c.$$

This, by collecting together the coefficients of $\sin x$ and $\cos x$ respectively, becomes

$$\sin x\left(1 - \frac{h^2}{1.2} + \frac{h^4}{1.2.3.4} - \&c.\right)$$

$$+ \cos x\left(h - \frac{h^3}{1.2.3} + \frac{h^5}{1.2.3.4.5} - \&c.\right),$$

which we know from (68) is equivalent to

$$\sin x \cos h + \cos x \sin h.$$

Similarly for the cosine.

Hence, if the sine and cosine of an arc were differentiated by means of a geometrical construction, or without taking for granted the expressions for $\sin(x+h)$ and $\cos(x+h)$, such expressions might be found from this example.

75. Cor. 1. The differential coefficients of every function of x being infinite in number unless it be of a rational algebraical form, it follows that the corresponding function of $x+h$ will in general be expressed by a series of terms indefinitely continued: and whenever the result of the application of this theorem is considered merely as an analytical transformation of the function, it is of very little consequence whether the number of terms be finite or not. This will not be the case however when we wish to ascertain the value of the function in any particular state of the principal variable, and

therefore it will be of some importance to be able to find the limit of the quantity which is omitted by stopping at any assigned term of the developement. We have seen that

$$u' = u + \frac{du}{dx}\,\frac{h}{1} + \frac{d^2u}{dx^2}\,\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\,\frac{h^2}{1.2.3} + \&c. + \frac{d^nu}{dx^n}\,\frac{h^3}{1.2.3.\&c.\,n}$$

$$+ \frac{d^{n+1}u}{dx^{n+1}}\,\frac{h^{n+1}}{1.2.3.\&c.\,(n+1)} + \frac{d^{n+2}u}{dx^{n+2}}\,\frac{h^{n+2}}{1.2.3.\&c.\,(n+2)}$$

$$+ \frac{d^{n+3}u}{dx^{n+3}}\,\frac{h^{n+3}}{1.2.3.\&c.\,(n+3)} + \&c.$$

whence if we stop at the end of the series comprised in the first line, we have the remainder equivalent to

$$\frac{h^{n+1}}{1.2.3.\&c.\,(n+1)}$$

$$\times \left\{ \frac{d^{n+1}u}{dx^{n+1}} + \frac{d^{n+2}u}{dx^{n+2}}\,\frac{h}{n+2} + \frac{d^{n+3}u}{dx^{n+3}}\,\frac{h^2}{(n+2)\,(n+3)} + \&c. \right\}$$

whereof the sum of the quantities contained between the brackets is obviously greater than its first term: but since by the theorem

$$\frac{d^{n+1}u'}{dx^{n+1}} = \frac{d^{n+1}u}{dx^{n+1}} + \frac{d^{n+2}u}{dx^{n+2}}\,\frac{h}{1} + \frac{d^{n+3}u}{dx^{n+3}}\,\frac{h^2}{1.2} + \&c.$$

of which each term after the first being manifestly greater than the corresponding term of the last mentioned series, since the the divisors of the quantity h and its powers are less, it follows that

$$\frac{d^{n+1}u}{dx^{n+1}} + \frac{d^{n+2}u}{dx^{n+2}}\,\frac{h}{n+2} + \frac{d^{n+3}u}{dx^{n+3}}\,\frac{h^2}{(n+2)\,(n+3)} + \&c.$$

is greater than $\dfrac{d^{n+1}u}{dx^{n+1}}$ and less than $\dfrac{d^{n+1}u'}{dx^{n+1}}$,

and therefore, that the value of the result given by *Taylor's* theorem lies between the two quantities,

$$u+\frac{du}{dx}\frac{h}{1}+\&c.+\frac{d^nu}{dx^n}\frac{h^n}{1.2.3.\&c.\,n}+\frac{d^{n+1}u}{dx^{n+1}}\frac{h^{n+1}}{1.2.3.\&c.\,(n+1)},$$

and

$$u+\frac{du}{dx}\frac{h}{1}+\&c.+\frac{d^nu}{dx^n}\frac{h^n}{1.2.3.\&c.\,n}+\frac{d^{n+1}u'}{dx^{n+1}}\frac{h^{n+1}}{1.2.3.\&c.\,(n+1)}:$$

and consequently the value of the remaining terms of the expansion after the $(n+1)^{\text{th}}$, lies between

$$\frac{d^{n+1}u}{dx^{n+1}}\frac{h^{n+1}}{1.2.3.\&c.\,(n+1)}\ \text{and}\ \frac{d^{n+1}u'}{dx^{n+1}}\frac{h^{n+1}}{1.2.3.\&c.\,(n+1)},$$

the least and greatest values of these differential coefficients corresponding to the limits x and $x+h$ of the independent variable being supposed to be taken.

Ex. 1. If we have $u=x^m$ and therefore $u'=(x+h)^m$, then will

$$\frac{d^{n+1}u}{dx^{n+1}}=m\,(m-1)\,(m-2)\,\&c.\,(m-n)\,x^{m-n-1},\ \text{and}$$

$$\frac{d^{n+1}u'}{dx^{n+1}}=m\,(m-1)\,(m-2)\,\&c.\,(m-n)\,(x+h)^{m-n-1};$$

whence, by substituting in the Theorem the values of the successive differential coefficients, it appears that the expansion of the binomial $(x+h)^m$ is comprised between the two expressions

$$x^m+mx^{m-1}h+\&c.+\frac{m\,(m-1)\,\&c.\,(m-n)}{1.2.3.\&c.\,(n+1)}x^{m-n-1}h^{n+1},$$

and

$$x^m+mx^{m-1}h+\&c.+\frac{m\,(m-1)\,\&c.\,(m-n)}{1.2.3.\&c.\,(n+1)}(x+h)^{m-n-1}h^{n+1}:$$

and the sum of the terms of the developement after the n^{th}, will obviously lie between

$$\frac{m\,(m-1)\,(m-2)\,\&c.\,(m-n+1)}{1.2.3.\&c.\,n}x^{m-n}h^n\ \text{and}$$

$$\frac{m\,(m-1)\,(m-2)\,\&c.\,(m-n+1)}{1.2.3.\&c.\,n}(x+h)^{m-n}h^n.$$

Ex. 2. Let $u=a^x$, so that $\frac{d^{n+1}u}{dx^{n+1}}=k^{n+1}a^x$ and $\frac{d^{n+1}u'}{dx^{n+1}}$ $=k^{n+1}a^{x+h}$: whence the developement of a^x is comprised between the values

$$a^x\left\{1+kh+\frac{k^2h^2}{1.2}+\&c.+\frac{k^{n+1}h^{n+1}}{1.2.3.\&c.(n+1)}\right\}$$

$$\text{and}\quad a^x\left\{1+kh+\frac{k^2h^2}{1.2}+\&c.+\frac{k^{n+1}h^{n+1}a^h}{1.2.3.\&c.(n+1)}\right\};$$

and if we stop at the n^{th} term of the developement, the sum of the remaining terms will obviously be greater than $\frac{k^nh^na^x}{1.2.3.\&c.\,n}$ and less than $\frac{k^nh^na^{x+h}}{1.2\;3.\&c.\,n}$.

Ex. 3. From $u=\log x$, we have immediately

$$\frac{d^{n+1}u}{dx^{n+1}}=-\frac{1.2.3.\&c.\,n}{x^{n+1}}(-1)^{n+1}\text{ and }\frac{d^{n+1}u'}{dx^{n+1}}=-\frac{1.2.3.\&c.\,n}{(x+h)^{n+1}}(-1)^{n+1};$$

from which we conclude that the expansion of $\log(x+h)$ lies between the expressions

$$\log x+\frac{h}{x}-\frac{1}{2}\left(\frac{h}{x}\right)^2+\&c.-\frac{1}{n}\left(-\frac{h}{x}\right)^n-\frac{1}{n+1}\left(-\frac{h}{x}\right)^{n+1},$$

and

$$\log x+\frac{h}{x}-\frac{1}{2}\left(\frac{h}{x}\right)^2+\&c.-\frac{1}{n}\left(-\frac{h}{x}\right)^n-\frac{1}{n+1}\left(-\frac{h}{x+h}\right)^{n+1};$$

and that the sum of the terms of the developement, after the n^{th}, is of intermediate magnitude to

$$-\frac{1}{n}\left(-\frac{h}{x}\right)^n\quad\text{and}\quad-\frac{1}{n}\left(-\frac{h}{x+h}\right)^n.$$

Ex. 4. Let $u = \sin x$; from which we obtain

$$\frac{d^{n+1}u}{dx^{n+1}} = \sin\left\{(n+1)\frac{\pi}{2} + x\right\}$$

and $$\frac{d^{n+1}u'}{dx^{n+1}} = \sin\left\{(n+1)\frac{\pi}{2} + x + h\right\};$$

and therefore the developement of $\sin(x+h)$ manifestly lies between the quantities

$$\sin x + \&\text{c.} + \sin\left(n\frac{\pi}{2} + x\right)\frac{h^n}{1.2.3.\&\text{c.}\,n} + \sin\left\{(n+1)\frac{\pi}{2} + x\right\}\frac{h^{n+1}}{1.2.3.\&\text{c.}\,(n+1)} \text{ and}$$

$$\sin x + \&\text{c.} + \sin\left(n\frac{\pi}{2} + x\right)\frac{h^n}{1.2.3.\&\text{c.}\,n} + \sin\left\{(n+1)\frac{\pi}{2} + x + h\right\}\frac{h^{n+1}}{1.2.3.\&\text{c.}\,(n+1)};$$

and consequently after n terms of the theorem have been applied, what remains will be intermediate to

$$\sin\left(n\frac{\pi}{2} + x\right)\frac{h^n}{1.2.3.\&\text{c.}\,n} \quad \text{and} \quad \sin\left(n\frac{\pi}{2} + x + h\right)\frac{h^n}{1.2.3.\&\text{c.}\,n};$$

and if the greatest and least values, 1 and -1, of the differential coefficient $\frac{d^n u}{dx^n}$ be taken, the magnitude of the sum of the remaining terms will, in all cases, lie between

$$-\frac{h^n}{1.2.3.\&\text{c.}\,n} \quad \text{and} \quad +\frac{h^n}{1.2.3.\&\text{c.}\,n}.$$

Similar conclusions may be drawn when $u = \cos x$; and the same method may be applied to every function whereof the general differential coefficient can be obtained.

76. Cor. 2. In the difference of the functions u and u', which is

$$u' - u = \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\frac{h^3}{1.2.3} + \&c.$$

any one of the terms may, by assigning a proper value to h, be made greater than the sum of all those that follow it.

For, if the expression be written $P_1 h + P_2 h^2 + \&c. + P_n h^n + \&c.$ and L be a quantity equal to, or greater than the greatest of the quantities P_1, P_2, P_3, &c. P_n, &c. then is

$$P_1 h + P_2 h^2 + P_3 h^3 + \&c. \text{ less than } Lh + Lh^2 + Lh^3 + \&c.$$

which will also manifestly be the case when

$$P_2 h^2 + P_3 h^3 + \&c. \text{ is less than } Lh^2 + Lh^3 + \&c.$$

$$\text{or less than } Lh^2 (1 + h + h^2 + \&c. \textit{ in infinitum})$$

$$\text{or less than } \frac{Lh^2}{1-h}:$$

wherefore, if $P_1 h$ be taken equal to, or greater than $\dfrac{Lh^2}{1-h}$;

$$\text{or } P_1 \text{ be taken equal to, or greater than } \frac{Lh}{1-h},$$

$$\text{or } h \text{ be taken equal to, or less than } \frac{P_1}{P_1 + L},$$

it is manifest that the first term will be greater than the sum of all the rest:

Again, after n terms, the developement is

$$P_{n+1} h^{n+1} + P_{n+2} h^{n+2} + \&c. = h^n (P_{n+1} h + P_{n+2} h^2 + \&c.)$$

in which $P_{n+1} h$ may be made greater than

$$P_{n+2} h^2 + P_{n+3} h^3 + \&c.,$$

by what has been proved; therefore it follows that on the same hypothesis

$$P_{n+1}h^{n+1} \text{ will be greater than } P_{n+2}h^{n+2}+P_{n+3}h^{n+3}+\text{\&c.}$$

Wherefore, if the inverse ratio of every two consecutive differential coefficients be finite, such a value may always be given to h, that any one of the terms may be greater than the sum of all those that succeed it.

77. Cor. 3. In every function of one principal variable, the general values of the successive differential coefficients will always be expressed in finite terms, except in certain rational functions, when they at length become $=0$; and it therefore follows, that *Taylor's* theorem will enable us to find the *general* developements of all functions of $x+h$ whatsoever.

But since in the demonstration of the theorem given in (74) the coefficients $P, Q, R,$ &c. of h and its powers have been supposed finite quantities, in order that the indices and coefficients may be respectively equated on both sides of the equation, the consequence will be, that if such a value be assigned to the principal variable as to make any of the said coefficients indefinitely great, the theorem cannot be expected any longer to hold good: in other words, it is said to *fail* in giving the developement; and this *Failure* therefore demonstrates, that the particular value of the function proposed cannot be expressed in *integral positive* powers of h combined with *finite* coefficients.

Ex. 1. Let $u=x^2+\sqrt{x-a}$ and $u'=(x+h)^2+\sqrt{x+h-a}$;

$$\therefore \frac{du}{dx}=2x+\tfrac{1}{2}(x-a)^{-\frac{1}{2}}, \quad \frac{d^2u}{dx^2}=2-\tfrac{1}{4}(x-a)^{-\frac{3}{2}},$$

$$\frac{d^3u}{dx^3}=\tfrac{3}{8}(x-a)^{-\frac{5}{2}}, \text{ \&c.}=\text{\&c.}$$

whence we have by the theorem,

$$u' = x^2 + \sqrt{x-a} + \left\{2x + \tfrac{1}{2}(x-a)^{-\frac{1}{2}}\right\}\frac{h}{1}$$

$$+ \left\{2 - \tfrac{1}{4}(x-a)^{-\frac{3}{2}}\right\}\frac{h^2}{1.2} + \left\{\tfrac{3}{8}(x-a)^{-\frac{5}{2}}\right\}\frac{h^3}{1.2.3} + \&c.$$

$$= x^2 + 2xh + h^2 + \sqrt{x-a} + \frac{h}{2(x-a)^{\frac{1}{2}}} - \frac{h^2}{8(x-a)^{\frac{3}{2}}} + \frac{h^3}{16(x-a)^{\frac{5}{2}}}$$
$- \&c.$

which is the *general* developement of u':

but its particular value corresponding to $x = a$, is

$$a^2 + 2ah + h^2 + 0 + \infty - \infty + \infty - \&c.$$

from which nothing can be determined, though at the same time the real value by substitution is $(a+h)^2 + \sqrt{h}$, which manifestly cannot be exhibited in integral positive powers of h with finite coefficients.

Ex. 2. Let $u = \dfrac{a}{(b-x)^3} = a(b-x)^{-3}$, and $\therefore\ u' = \dfrac{a}{(b-x-h)^3}$:

$$\text{then } \frac{du}{dx} = 3a(b-x)^{-4},\quad \frac{d^2u}{dx^2} = 3.4\,a(b-x)^{-5},$$

$$\frac{d^3u}{dx^3} = 3.4.5\,a(b-x)^{-6},\ \&c. = \&c.$$

whence the *general* developement will be

$$u' = \frac{a}{(b-x)^3} + \frac{3a}{(b-x)^4}\,\frac{h}{1} + \frac{3.4\,a}{(b-x)^5}\,\frac{h^2}{1.2} + \frac{3.4.5\,a}{(b-x)^6}\,\frac{h^3}{1.2.3} + \&c.$$

in which for the particular case of $x = b$, every term becomes infinite, though the real value of the function is then $= -\dfrac{a}{h^3} = -ah^{-3}$, the index of h being negative.

78. Cor. 4. Though fractional and negative indices may enter into the developement of a function when particular values are assigned to the independent variable, it may readily be demonstrated *a priori* that such cannot be the case whilst x remains indeterminate.

For, if possible, let $Gh^{\frac{m}{n}}$ or $G\sqrt[n]{h^m}$ be one of the terms of the general developement of $f(x+h)$, then it is obvious that the radical quantities which are involved in fx will also be found with the same indices in $f(x+h)$: now the substitution of $x+h$ in the place of x, cannot alter the number of the values of the function, it being observed that every radical quantity has as many different values as there are units in its exponent, and that every irrational function has therefore as many different values as there can be formed combinations of the different values of the radical quantities which it contains; and it therefore necessarily follows that if the developement contained a term of the form $G\sqrt[n]{h^m}$, every value of fx would combine itself with the n values of the radical quantity $\sqrt[n]{h^m}$, so that the developed function would comprise more values than the same function when not developed; which is absurd.

Neither can the developement involve a term of the form Gh^{-m}, or $\frac{G}{h^m}$, for when $h=0$, $\frac{G}{h^m}=\infty$, whence we should have $fx=\infty$, which is contrary to the hypothesis.

It therefore remains only that the developement of $f(x+h)$ is of the form

$$fx + Ph + Qh^2 + Rh^3 + \&c.$$

which has evidently as many values as fx, there being a value of each of the quantities P, Q, R, &c. corresponding to each value of the function proposed.

79. Cor. 5. From what has been said then, it is obvious that the same conclusion will no longer hold good when such

a value of the principal variable is substituted for it in the function as will cause to disappear from it one or more radical quantities: for though such particular value may cause the radical quantities to disappear from fx, it will not have the same effect upon $f(x+h)$; and the developement of $f(x+h)$ may, for this particular value of x, contain more values than the original function of x, which excess is the cause of fractional powers of h being admitted into the developement.

80. Cor. 6. Conversely, if the developement of $f(x+h)$, for any particular value of the independent variable, involve either fractional or negative coefficients, there is no difficulty in shewing that some or all of the corresponding differential coefficients ought to become infinite.

For, when the value a is assigned to x, let us assume

$$f(a+h) = A_0 + A_1 h + A_2 h^2 + \&c. + A_n h^n + A_\alpha h^\alpha + \&c.$$

where α is a fraction lying between the whole numbers n and $n+1$: then we know that A_0, A_1, &c. A_n, &c. are the values of the function and its successive differential coefficients when $x = a$: but since, when

$$u' = f(x+h), \text{ we know that } \frac{du'}{dx} = \frac{du'}{dh},$$

we shall obviously have

$$A_0 = f(a+h), \quad A_1 = \frac{df(a+h)}{dh}, \quad A_2 = \frac{d^2 f(a+h)}{dh^2}, \quad \&c. = \&c.$$

$$A_n = \frac{d^n f(a+h)}{dh^n}, \text{ the value of } h \text{ being taken } = 0:$$

now from the assumption, we have

$$\frac{d^n f(a+h)}{dh^n} = 1.2.3.\&c.\, n A_n + \alpha(\alpha-1)(\alpha-2)\,\&c.$$
$$(\alpha - n + 1) A_\alpha h^{\alpha - n} + \&c.$$

$$\frac{d^{n+1} f(a+h)}{dh^{n+1}} = \alpha(\alpha-1)(\alpha-2)\,\&c.\,(\alpha-n) A_\alpha h^{\alpha-n-1} + \&c.$$

$$\&c. = \&c. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

wherefore, if $h=0$, it is obvious that A_n remains finite, whilst A_α and the coefficients succeeding it, become infinite.

Again, if we assume

$$f(a+h)=A_{-m}h^{-m}+\&c.+A_0+A_1h+A_2h^2+\&c.$$

we shall immediately obtain

$$\frac{df(a+h)}{dh}=-mA_{-m}h^{-m-1}+\&c.+A_1+2A_2h+\&c.$$

$$\frac{d^2f(a+h)}{dh^2}=m(m+1)A_{-m}h^{-m-2}+\&c.+2A_2+\&c.$$

$$\&c.=\&c.\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

which, on the supposition of h being $=0$, all become infinite.

Hence also, if the first n values of the differential coefficients of $f(a+h)$ remain finite, and all afterwards become infinite, it follows that a term of the form $A_\alpha h^\alpha$, where α lies between n and $n+1$, must enter into the corresponding developement of the function: and though all the preceding terms of the developement are correctly found, and its place has thus been ascertained, the term itself must be determined by other methods, which do not depend upon this Calculus.

81. *Taylor's* theorem, proved in the preceding pages, gives

$$u'=u+\frac{du}{dx}\frac{h}{1}+\frac{d^2u}{dx^2}\frac{h^2}{1.2}+\frac{d^3u}{dx^3}\frac{h^3}{1.2.3}+\&c.$$

$$\therefore\ u'-u \text{ or } \Delta u=\frac{du}{dx}\frac{h}{1}+\frac{d^2u}{dx^2}\frac{h^2}{1.2}+\frac{d^3u}{dx^3}\frac{h^3}{1.2.3}+\&c.$$

which is the value of the increment of u corresponding to h, the increment of the principal variable x; and if h be assumed equal to the indeterminate magnitude dx, the expression becomes

$$\Delta u = \frac{du}{1} + \frac{d^2u}{1.2} + \frac{d^3u}{1.2.3} + \&c. + \frac{d^nu}{1.2.3.\&c.n} + \&c.$$

which is the increment or finite difference of the function expressed in terms of its successive differentials.

Ex. Let $u = ax^m$, then the formula above gives

$$\Delta u = d(ax^m) + \frac{1}{1.2}d^2(ax^m) + \frac{1}{1.2.3}d^3(ax^m) + \&c.$$

$$\text{but } d(ax^m) = max^{m-1}dx;$$

$$d^2(ax^m) = d(max^{m-1}dx) = m(m-1)ax^{m-2}dx^2;$$

$$d^3(ax^m) = d\{m(m-1)ax^{m-2}dx^2\} = m(m-1)(m-2)ax^{m-3}dx^3;$$

$$\&c. = \&c.$$

whence we have

$$\Delta u = max^{m-1}dx + \frac{m(m-1)}{1.2}ax^{m-2}dx^2$$

$$+ \frac{m(m-1)(m-2)}{1.2.3}ax^{m-3}dx^3 + \&c.$$

82. The expression for the finite difference of u exhibited in terms of its successive differentials, will therefore enable us to find the general differential coefficient of any proposed function fx, provided the general term of the developement of $f(x+dx)$ can be determined.

For, $\Delta u = f(x+dx) - fx$

$$= fx + Pdx + Qdx^2 + Rdx^3 + \&c. + Xdx^n + \&c. - fx$$

$$= Pdx + Qdx^2 + Rdx^3 + \&c. + Xdx^n + \&c.:$$

$$\text{but } \Delta u = \frac{du}{1} + \frac{d^2u}{1.2} + \frac{d^3u}{1.2.3} + \&c. + \frac{d^nu}{1.2.3.\&c.n} + \&c.$$

$\therefore$ equating the corresponding terms of these two series, we have

$$du = Pdx,\ \frac{d^2u}{1.2} = Qdx^2,\ \frac{d^3u}{1.2.3} = Rdx^3,\ \text{\&c.} = \text{\&c.}$$

$$\frac{d^nu}{1.2.3.\text{\&c.}\,n} = Xdx^n:$$

whence we have $\dfrac{du}{dx} = P,\ \dfrac{d^2u}{dx^2} = 1.2Q.$

$$\frac{d^3u}{dx^3} = 1.2.3R,\ \text{\&c.} = \text{\&c.},\ \frac{d^nu}{dx^n} = 1.2.3.\text{\&c.}\,nX.$$

Ex. 1. Let $u = x^m$, therefore by the binomial theorem we have

$$\Delta\, u = (x+dx)^m - x^m = mx^{m-1}dx$$

$$+ \frac{m(m-1)}{1.2}x^{m-2}dx^2 + \frac{m(m-1)(m-2)}{1.2.3}x^{m-3}dx^3 + \text{\&c.}$$

$$+ \frac{m(m-1)(m-2)\ \text{\&c.}\ (m-n+1)}{1.2.3.\text{\&c.}\,n}x^{m-n}dx^n + \text{\&c.}$$

$$= \frac{du}{1} + \frac{d^2u}{1.2} + \frac{d^3u}{1.2.3} + \text{\&c.} + \frac{d^nu}{1.2.3.\text{\&c.}\,n} + \text{\&c.}$$

from which we obtain

$$\frac{du}{dx} = mx^{m-1};\ \frac{d^2u}{dx^2} = m(m-1)x^{m-2};$$

$$\frac{d^3u}{dx^3} = m(m-1)(m-2)x^{m-3};\ \text{\&c.} = \text{\&c.}$$

$$\frac{d^nu}{dx^n} = m(m-1)(m-2)\ \text{\&c.}\ (m-n+1)x^{m-n}.$$

Ex. 2. Let $u = (a+bx+cx^2)^m$, therefore

$$\Delta\, u = (a+bx+cx^2+bdx+2cxdx+cdx^2)^m - (a+bx+cx^2)^m$$

$$= (p+qdx+cdx^2)^m - p^m,$$

if $a+bx+cx^2 = p$, and $b+2cx = q$:

but, by the binomial theorem,

$$(p+qdx+cdx^2)^m=(p+qdx)^m+m\,(p+qdx)^{m-1}cdx^2$$

$$+\frac{m\,(m-1)}{1\,.\,2}(p+qdx)^{m-2}c^2dx^4$$

$$+\frac{m\,(m-1)\,(m-2)}{1\,.\,2\,.\,3}(p+qdx)^{m-3}c^3dx^6+\&c.$$

and to obtain the coefficient of dx^n, we must manifestly add together, in connection with constant quantities,

the coefficient P_n of dx^n in the developement of $(p+qdx)^m$,
.... P_{n-2} of dx^{n-2}.................... $(p+qdx)^{m-1}$,
.... P_{n-4} of dx^{n-4}.................... $(p+qdx)^{m-2}$,
.... P_{n-6} of dx^{n-6}.................... $(p+qdx)^{m-3}$,
&c..............................&c..............................

$$\text{hence } \frac{d^nu}{1.2.3.\,\&c.\,n}=\Big\{P_n+mP_{n-2}c+\frac{m\,(m-1)}{1.2}P_{n-4}c^2$$

$$+\frac{m(m-1)(m-2)}{1.2.3}P_{n-6}c^3+\&c.\Big\}dx^n,$$

$$\text{and } d^nu=1.2.3.\&c.\,n\,\Big\{P_n+mP_{n-2}c+\frac{m\,(m-1)}{1.2}P_{n-4}c^2$$

$$+\frac{m\,(m-1)\,(m-2)}{1.2.3}P_{n-6}c^3+\&c.\Big\}dx^n:$$

but, from the binomial theorem, we readily obtain

$$P_n=\frac{m\,(m-1)\,(m-2)\,\&c.\,(m-n+1)}{1\,.\,2\,.\,3\,.\,\&c.\,n}p^{m-n}q^n;$$

$$P_{n-2}=\frac{(m-1)\,(m-2)\,(m-3)\,\&c.\,(m-n+2)}{1\,.\,2\,.\,3\,.\,\&c.\,(n-2)}p^{m-n+1}q^{n-2}$$

$$=P_n\frac{n\,(n-1)}{m\,(m-n+1)}\,\frac{p}{q^2};$$

$$P_{n-4} = \frac{(m-2)(m-3)(m-4)\,\&c.\,(m-n+3)}{1.2.3.\&c.\,(n-4)} p^{m-n+2} q^{n-4}$$

$$= P_n \frac{n(n-1)(n-2)(n-3)}{m(m-1)(m-n+1)(m-n+2)} \frac{p^2}{q^4};$$

$$P_{n-6} = \frac{(m-3)(m-4)(m-5)\,\&c.\,(m-n+4)}{1.2.3.\&c.\,(n-6)} p^{m-n+3} q^{n-6}$$

$$= P_n \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{m(m-1)(m-2)(m-n+1)(m-n+2)(m-n+3)} \frac{p^3}{q^6};$$

&c. = &c...

therefore, by substitution and reduction, we get

$$\frac{d^n u}{dx^n} = m(m-1)(m-2)\,\&c.\,(m-n+1)\,p^{m-n} q^n$$

$$\times \left\{ 1 + \frac{n(n-1)}{(m-n+1)} \frac{cp}{q^2} + \frac{n(n-1)(n-2)(n-3)}{(m-n+1)(m-n+2)} \frac{c^2 p^2}{1.2\,q^4} \right.$$

$$\left. + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{(m-n+1)(m-n+2)(m-n+3)} \frac{c^3 p^3}{1.2.3\,q^6} + \&c. \right\},$$

in which p and q must be replaced by their values

$$(a + bx + cx^2) \text{ and } (b + 2cx).$$

This example includes a greater variety of others, formed by assigning different values to the quantities a, b, c, and which may be done out in full for practice.

83. *Given the differential coefficients belonging to the equation* u = f x, *to find the differential coefficients belonging to the corresponding equation* x = ϕu.

Let h and k be the contemporaneous indeterminate increments of x and u; then, if p, q, r, &c. denote the differential coefficients of u, from $u = fx$ we shall have

$$k = p\frac{h}{1} + q\frac{h^2}{1.2} + r\frac{h^3}{1.2.3} + \&c.$$

and if p', q', r', &c. be the differential coefficients of x, from $x = \phi u$ we get

$$h = p'\frac{k}{1} + q'\frac{k^2}{1.2} + r'\frac{k^3}{1.2.3} + \&c.$$

$$= p'\left\{p\frac{h}{1} + q\frac{h^2}{1.2} + r\frac{h^3}{1.2.3} + \&c.\right\}$$

$$+ \frac{q'}{1.2}\left\{p\frac{h}{1} + q\frac{h^2}{1.2} + r\frac{h^3}{1.2.3} + \&c.\right\}^2$$

$$+ \frac{r'}{1.2.3}\left\{p\frac{h}{1} + q\frac{h^2}{1.2} + r\frac{h^3}{1.2.3} + \&c.\right\}^3 + \&c.$$

$$= pp'h + (p'q + q'p^2)\frac{h^2}{1.2} + (p'r + 3pqq' + r'p^3)\frac{h^3}{1.2.3} + \&c.$$

whence, equating corresponding coefficients, we obtain

$$pp' = 1, \text{ and } \therefore\ p' = \frac{1}{p}, \text{ or } \frac{dx}{du} = \frac{1}{\left(\frac{du}{dx}\right)};$$

$$p'q + q'p^2 = 0, \text{ and } \therefore\ q' = -\frac{qp'}{p^2} = -\frac{q}{p^3},$$

$$\text{or } \frac{d^2x}{du^2} = -\frac{\left(\frac{d^2u}{dx^2}\right)}{\left(\frac{du}{dx}\right)^3};$$

$$p'r + 3pqq' + r'p^3 = 0, \text{ and } \therefore\ r' = \frac{3q^2 - pr}{p^5},$$

$$\text{or } \frac{d^3x}{du^3} = 3\frac{\left(\frac{d^2u}{dx^2}\right)^2}{\left(\frac{du}{dx}\right)^5} - \frac{\left(\frac{d^3u}{dx^3}\right)}{\left(\frac{du}{dx}\right)^4}; \text{ and so on.}$$

Ex. To illustrate this article in the function $u = \tan x$, we have

$$\frac{du}{dx} = (\sec x)^2, \quad \frac{d^2u}{dx^2} = 2 \tan x\, (\sec x)^2, \text{ \&c.}$$

whence are obtained

$$\frac{dx}{du} = \frac{1}{(\sec x)^2} = \frac{1}{1+u^2};$$

$$\frac{d^2x}{du^2} = -\frac{2 \tan x\, (\sec x)^2}{(\sec x)^6} = -\frac{2 \tan x}{(\sec x)^4} = -\frac{2u}{(1+u^2)^2}, \text{ \&c.}$$

as has also been proved in a preceding article.

84. *If* u *be considered a function of* y *at the same time that* y *is considered a function of* x, *it is required to find the differential coefficients of* u *considered a function of* x.

Let $u = Fy$ and $y = fx$, then by virtue of these two equations we must obviously have $u = \phi x$:

now, if h and k denote the contemporaneous increments of x and y, we shall have from $y = fx$ and $u = \phi x$,

$$k = \frac{dy}{dx} h + \frac{d^2y}{dx^2} \frac{h^2}{1.2} + \frac{d^3y}{dx^3} \frac{h^3}{1.2.3} + \text{\&c.}$$

$$\text{and } u' - u = \frac{du}{dx} h + \frac{d^2u}{dx^2} \frac{h^2}{1.2} + \frac{d^3u}{dx^3} \frac{h^3}{1.2.3} + \text{\&c.}:$$

but from $u = Fy$, we likewise have

$$u' - u = \frac{du}{dy} k + \frac{d^2u}{dy^2} \frac{k^2}{1.2} + \frac{d^3u}{dy^3} \frac{k^3}{1.2.3} + \text{\&c.}$$

which, by substitution for k and its powers, becomes

$$\frac{du}{dy} \left\{ \frac{dy}{dx} h + \frac{d^2y}{dx^2} \frac{h^2}{1.2} + \text{\&c.} \right\}$$

$$+\frac{d^2u}{1.2dy^2}\left\{\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\&c.\right\}^2$$

$$+\frac{d^3u}{1.2.3dy^3}\left\{\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\&c.\right\}^3$$

$$+\&c. \ldots\ldots\ldots\ldots\ldots\ldots$$

whence equating the coefficients of the same powers of h in the two expressions for $u'-u$, we obtain

$$\frac{du}{dx}=\frac{du}{dy}\times\frac{dy}{dx}:$$

$$\frac{d^2u}{dx^2}=\frac{du}{dy}\frac{d^2y}{dx^2}+\frac{d^2u}{dy^2}\frac{dy^2}{dx^2}:$$

$$\&c. = \&c. \ldots\ldots\ldots\ldots$$

85. Cor. From the equations just found are readily deduced

$$\frac{du}{dy}=\frac{\frac{du}{dx}}{\frac{dy}{dx}}.$$

$$\frac{d^2u}{dy^2}=\frac{\frac{d^2u}{dx^2}-\frac{du}{dy}\frac{d^2y}{dx^2}}{\frac{dy^2}{dx^2}}=\frac{\frac{dy}{dx}\frac{d^2u}{dx^2}-\frac{du}{dx}\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}: \&c.,$$

and similarly the other differential coefficients taken with respect to y may be expressed in terms of those taken with respect to x.

Ex. 1. Let $u=\sin y$, where $y=\cos^{-1}x$,

$$\text{then } \frac{du}{dy}=\cos y \text{ and } \frac{dy}{dx}=-\frac{1}{\sqrt{1-x^2}};$$

$$\therefore \frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx} = -\frac{\cos y}{\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}},$$

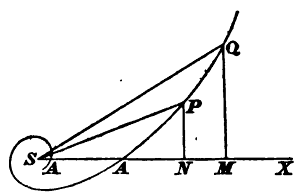
as would readily appear from the equation

$$u = \sin\cos^{-1} x = \sqrt{1-x^2}.$$

Ex. 2. In article (56) we have seen that according to the notation there used

$$ds = \sqrt{dx^2 + dy^2}.$$

Wherefore if in the annexed figure r represent the radius vector SP and θ be the angle which it makes with the axis SX,



the rectangular co-ordinates SN and NP being denoted by x and y respectively, we shall have

$$x = r\cos\theta \text{ and } y = r\sin\theta:$$

but by virtue of the equation $r = f\theta$, it is obvious that both x and y are functions of θ; therefore

$$\frac{ds}{d\theta} = \sqrt{\frac{dx^2}{d\theta^2} + \frac{dy^2}{d\theta^2}}:$$

$$\text{now } \frac{dx}{d\theta} = -r\sin\theta + \cos\theta\frac{dr}{d\theta},$$

$$\frac{dy}{d\theta} = r\cos\theta + \sin\theta\frac{dr}{d\theta};$$

whence by substitution, we obtain

$$\frac{ds}{d\theta} = \sqrt{\left(\cos\theta\frac{dr}{d\theta} - r\sin\theta\right)^2 + \left(\sin\theta\frac{dr}{d\theta} + r\cos\theta\right)^2}$$

$$= \sqrt{\frac{dr^2}{d\theta^2} + r^2}, \text{ and } \therefore\ ds = \sqrt{dr^2 + r^2 d\theta^2}.$$

Again, $dS = dASP = d(SPN - APN)$

$$= d\left(\frac{xy}{2}\right) - ydx = \tfrac{1}{2}(xdy - ydx), \text{ by (57)};$$

$$\therefore\ \frac{dS}{d\theta} = \tfrac{1}{2}\left\{x\frac{dy}{d\theta} - y\frac{dx}{d\theta}\right\} = \frac{r^2}{2}, \text{ and thence } dS = \frac{r^2 d\theta}{2}.$$

Ex. 3. If $u = a(x + \sin x)$ and $y = a(1 - \cos x)$, then will

$$\frac{du}{dx} = a(1 + \cos x) \text{ and } \frac{dy}{dx} = a\sin x;$$

whence, by means of the formulæ just investigated, we obtain

$$\frac{du}{dy} = \frac{\frac{du}{dx}}{\frac{dy}{dx}} = \frac{1 + \cos x}{\sin x}:$$

$$\text{again, } \frac{d^2u}{dx^2} = -a\sin x \text{ and } \frac{d^2y}{dx^2} = a\cos x;$$

$$\therefore\ \frac{d^2u}{dy^2} = \frac{-(a\sin x)^2 - a(1 + \cos x)a\cos x}{a^3(\sin x)^3} = -\frac{1 + \cos x}{a(\sin x)^3};$$

and so on: and it may here be observed that it would have been impossible first to eliminate x and its trigonometrical functions and then to have differentiated immediately with respect to y.

85. In the equation of *Taylor's* Theorem found in (74),

$$u' = u + \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\frac{h^3}{1.2.3} + \&c.$$

if we suppose $x = 0$, it is evident that u' then becomes the same function of h, that u is of x; and therefore, representing the corresponding values of

$$u, \frac{du}{dx}, \frac{d^2u}{dx^2}, \frac{d^3u}{dx^3}, \text{ \&c. by } U_0, U_1, U_2, U_3, \text{ \&c. respectively,}$$

we shall have

$$f(h) = U_0 + U_1 \frac{h}{1} + U_2 \frac{h^2}{1.2} + U_3 \frac{h^3}{1.2.3} + \text{\&c.}:$$

whence by changing h into x in both members, we get

$$f(x) \text{ or } u = U_0 + U_1 \frac{x}{1} + U_2 \frac{x^2}{1.2} + U_3 \frac{x^3}{1.2.3} + \text{\&c.}$$

which is *Maclaurin's* Theorem investigated in article (69).

86. In the investigations of all the rules and formulæ contained and applied in the preceding Chapters, the ratios of the increments or finite differences of functions, and of the variable quantities on which they depend, have been expressed in terms of their original values and indeterminate increments, and thence the ultimate values of these ratios have been deduced by the method of *Limits*, exactly in the same manner as *Newton* has done in the different sections of the *Principia*.

This method of proceeding does not essentially differ from that of *D'Alembert*, in which if

$$u = f(x) \text{ and } u' = f(x + h),$$

he proposes to determine the value of the fraction $\frac{u' - u}{h}$ when the quantity h becomes evanescent.

It is manifest that the sole difficulty in the method here pursued, is the determining of the value of a ratio, when the terms, which express it, tend continually to a state of

evanescence; but it is also obvious that this difficulty will in a great degree disappear, when it is considered that the magnitude of a ratio depends not upon the absolute, but relative magnitudes of its terms, and that this ratio has in all cases been exhibited by means of finite and determinate quantities.

Thus have we been brought to the conclusion, that, if u be a function of x, and u' the same function of $x+h$, the general relation subsisting among the quantities u, u', x and h will be expressed by the equation

$$u' = u + \frac{du}{dx}\,\frac{h}{1} + \frac{d^2u}{dx^2}\,\frac{h^2}{1\,.\,2} + \frac{d^3u}{dx^3}\,\frac{h^3}{1\,.\,2\,.\,3} + \&c.$$

$$\text{and } \therefore\ u' - u = \frac{du}{dx}\,\frac{h}{1} + \frac{d^2u}{dx^2}\,\frac{h^2}{1\,.\,2} + \frac{d^3u}{dx^3}\,\frac{h^3}{1\,.\,2\,.\,3} + \&c.$$

in which the coefficients of $\frac{h}{1}$, $\frac{h^2}{1\,.\,2}$, $\frac{h^3}{1\,.\,2\,.\,3}$, &c. are the differential coefficients of u, whose orders are the corresponding indices of h.

The conclusions, at which this Calculus enables us to arrive, are however deducible from several other views of the subject, which shall now be briefly explained.

I. FLUXIONS.

87. In *Sir Isaac Newton's* method of *Fluxions*, all quantities whatever are supposed to be generated by continuous motion: as for instance, lines by the motions of points, surfaces by the motions of lines, solids by the motions of surfaces, &c.

The relative rates or velocities, wherewith dependent magnitudes or functions, and the principal variables on which they depend, are increasing at any instant, are termed their *Fluxions*, and the whole quantities generated in consequence of such velocities of increase, the *Fluents*.

These velocities or Fluxions are represented by the same letters as the quantities themselves with points or dots placed over them: thus, if u and x represent two connected quantities generated by continuous motion, $\dot{u}$ and $\dot{x}$ are taken to denote the *rates* of their increase at any time, and therefore $\frac{\dot{u}}{\dot{x}}$ will express the *relation* between the rates at which their increments are generated.

Also, since the numerical magnitudes of all quantities whatever may be represented by straight lines, the *Moments* or indefinitely small portions of the quantities generated will be equal to the products of their velocities and the indefinitely small portion of time during which these velocities are continued, so that, if τ denote this indefinitely small portion of time, the moments of u and x will be represented by $\tau\dot{u}$ and $\tau\dot{x}$ respectively.

What has been here said, will become more intelligible by the perusal of the following examples.

Ex. 1. Let $u = ax^m$, then since $\dot{u}$ and $\dot{x}$ represent the velocities with which u and x are respectively increasing, their moments will be $\tau\dot{u}$ and $\tau\dot{x}$:

now the proposed equation $u = ax^m$ being true in all its states, we shall consequently have

$$\therefore\ u + \tau\dot{u} = a(x + \tau\dot{x})^m$$

$$= ax^m + max^{m-1}\tau\dot{x} + \frac{m(m-1)}{1.2}ax^{m-2}\tau^2\dot{x}^2 + \&c.:$$

$$\text{hence } \tau\dot{u} = max^{m-1}\tau\dot{x} + \frac{m(m-1)}{1.2}ax^{m-2}\tau^2\dot{x}^2 + \&c.$$

$$\text{and } \therefore\ \dot{u} = max^{m-1}\dot{x} + \frac{m(m-1)}{1.2}ax^{m-2}\tau\dot{x}^2 + \&c.$$

$$=max^{m-1}\dot{x}, \text{ since } \tau \text{ is of evanescent magnitude:}$$

also, because the binomial Theorem is general, this will be true whether m is positive or negative, integral or fractional.

Ex. 2. Let $u=\sin x$, then $u+\tau\dot{u}=\sin(x+\tau\dot{x})$,

$$\text{and } \therefore\ \tau\dot{u}=\sin(x+\tau\dot{x})-\sin x=2\cos\left(\frac{2x+\tau\dot{x}}{2}\right)\sin\frac{\tau\dot{x}}{2};$$

$$\text{whence we have } \dot{u}=\cos\tfrac{1}{2}(2x+\tau\dot{x})\frac{\sin\frac{1}{2}\tau\dot{x}}{\frac{1}{2}\tau}=\cos x\dot{x},$$

since τ is indefinitely small, and therefore $\sin\dfrac{\tau\dot{x}}{2}=\dfrac{\tau\dot{x}}{2}$.

Ex. 3. Let $u=xy$, then $\dot{u}$, $\dot{x}$ and $\dot{y}$ denote the velocities with which u, x and y are respectively increasing, and their moments will therefore be $\tau\dot{u}$, $\tau\dot{x}$ and $\tau\dot{y}$:

$$\begin{aligned}\text{hence } u+\tau\dot{u}&=(x+\tau\dot{x})(y+\tau\dot{y})\\ &=xy+y\tau\dot{x}+x\tau\dot{y}+\tau^2\dot{x}\dot{y};\end{aligned}$$

$$\text{and } \therefore\ \tau\dot{u}=y\tau\dot{x}+x\tau\dot{y}+\tau^2\dot{x}\dot{y},$$

whence we have $\dot{u}=y\dot{x}+x\dot{y}$, since τ is indefinitely small.

Here we may observe that x and y are any quantities whatever.

Ex. 4. Let $u=xyz$, then, as in the preceding examples,

$$u+\tau\dot{u}=(x+\tau\dot{x})(y+\tau\dot{y})(z+\tau\dot{z})$$

$$=xyz+\tau(yz\dot{x}+xz\dot{y}+xy\dot{z})+\tau^2(x\dot{y}\dot{z}+y\dot{x}\dot{z}+z\dot{x}\dot{y})+\tau^3\dot{x}\dot{y}\dot{z};$$

whence is obtained $\dot{u}=yz\dot{x}+xz\dot{y}+xy\dot{z}$, as before:

and similarly if there be more factors.

Ex. 5. Let s be the arc of a curve whose rectangular co-ordinates are x and y; then if its equation be $y=f(x)$,

the velocities of the generating point in the directions of x and y may at any time be represented by $\dot{x}$ and $\dot{y}$, and consequently the corresponding velocity in the direction of the curve will be that which is compounded of these two, that is

$$\dot{s} = \sqrt{\dot{x}^2 + \dot{y}^2}.$$

Hence, taking, as an instance, a circle whose equation is $y = \sqrt{2ax - x^2}$, we have by the first example,

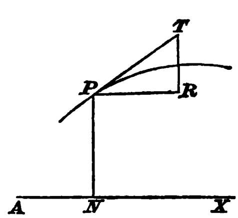
$$\dot{y} = \frac{(a-x)\dot{x}}{\sqrt{2ax-x^2}}, \text{ or } \frac{\dot{y}}{\dot{x}} = \frac{a-x}{\sqrt{2ax-x^2}};$$

and therefore

$$\dot{s} = \frac{a\dot{x}}{\sqrt{2ax-x^2}}, \text{ or } \frac{\dot{s}}{\dot{x}} = \frac{a}{\sqrt{2ax-x^2}}:$$

that is, the velocity in y is $\frac{a-x}{\sqrt{2ax-x^2}}$ times as great, and the velocity in s is $\frac{a}{\sqrt{2ax-x^2}}$ times as great as the velocity in x; also at any point (x, y) these velocities are to one another respectively as the quantities $a-x$, a and $\sqrt{2ax-x^2}$, which, by assigning numerical values to x, will give their corresponding numerical relations.

Since, by Lemma 7, *Newton's* Principia, the directions of the arc and tangent are ultimately coincident, if



at the point P a tangent of any length be supposed to be drawn, and the triangle PTR be completed,

the sides PR, RT and PT of the triangle

will be respectively proportional to the fluxions of the lines AN, NP and AP, denoted by x, y and s, and may consequently represent the finite velocities designated by $\dot{x}$, $\dot{y}$ and $\dot{s}$: and PTR is therefore called a *Fluxional* triangle.

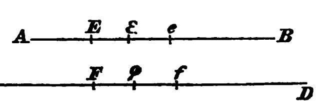
Ex. 6. Let Σ be the surface of a solid symmetrical with respect to its axis, and generated by the motion of a variable circle always parallel to itself; then if x and y be the abscissa and ordinate corresponding to the arc s of a section made by a plane drawn through the axis, it is evident that the moment of the surface of the solid : the moment of the surface of a cylinder having the same axis and the radius of whose base is y

:: the velocity in s : the velocity in x :: $\dot{s}$: $\dot{x}$;

whence we shall have $\dot{\Sigma} = 2\pi y \dot{x} \dfrac{\dot{s}}{\dot{x}} = 2\pi y \dot{s}$.

88. With respect to these examples, we observe that the results are the same as have already been obtained by the rules of the Differential Calculus laid down in the preceding Chapters; and it is not difficult to prove generally, that the limits of the ratios of the increments of the quantities there used are precisely the same as the ratios of the Fluxions above defined.

For, let AB and CD be two straight lines described with



continually accelerated velocities, Ee, Ff being simultaneous increments: let $E\epsilon$, $F\phi$ be the increments which would have been generated in the same time with the velocities at E and F continued constant: suppose also v to be the uniform velocity

with which $E\epsilon$ is described, $\dot{v}+\dot{w}$ the velocity acquired at e, and $v+z$ the uniform velocity with which Ee would have been described in the same time: then, it is manifest that $v+z$ is less than $v+w$ and greater than v, and therefore that z is less than w, and greater than zero; likewise when the increments are diminished *sine limite*, w and therefore z is also diminished *sine limite*; hence the limit of $E\epsilon : Ee = 1 : 1 =$ the limit of $F\phi : Ff$; and

$$\therefore \text{ the limit of } Ee : Ff = E\epsilon : F\phi:$$

but $E\epsilon$, $F\phi$ are the increments generated uniformly and may therefore be taken to represent the velocities or Fluxions:

whence the limit of $Ee : Ff =$ Fluxion of AE : Fluxion of CF.

Therefore, the rules previously investigated for finding the *Differentials* of quantities, will also be sufficient for determining their *Fluxions*.

Ex. 1. Let $u = ax^2 + bx + c$, then $\dot{u} = 2ax\dot{x} + b\dot{x}$, and $\frac{\dot{u}}{\dot{x}} = 2ax + b$, which shews that at any time the function u is increasing $2ax + b$ times as fast as the principal variable x.

Ex. 2. Let $u = a^x$, then by article (34) we have

$$\dot{u} = ka^x\dot{x}, \text{ and thence } \frac{\dot{u}}{\dot{x}} = ka^x;$$

that is, the ratio of the rates of increase of a simple exponential function and its index, is measured by the function multiplied by a constant quantity; and also, if x increase in *Arithmetical* progression, it appears that this ratio increases in *Geometrical*.

Ex. 3. Let $u = \log x$, then, in the *Napierian* System we have by (41)

$$\dot{u} = \frac{\dot{x}}{x}, \text{ and therefore } \frac{\dot{u}}{\dot{x}} = \frac{1}{x};$$

or the ratio of the rates of increase of the *Napierian* Logarithm of any quantity and of the quantity itself, is measured by the reciprocal of that quantity; and consequently is the less as the quantity is the greater.

Hence if the difference of two quantities be given, the difference of their logarithms will decrease as the quantities themselves increase.

Ex. 4. Let $u = \tan^{-1}x$, then by (52), we have

$$\dot{u} = \frac{\dot{x}}{1+x^2}, \text{ and } \therefore \frac{\dot{u}}{\dot{x}} = \frac{1}{1+x^2},$$

which expresses the ratio of the rates at which the circular arc and its Trigonometrical tangent are increasing.

Also, if $u = 45^0$, we have $x = 1$, and $\therefore \frac{\dot{u}}{\dot{x}} = \frac{1}{1+1} = \frac{1}{2}$,

which proves that the tangent of an arc of 45^0 is increasing twice as fast as the arc itself.

89. In the application of this Calculus, the Fluxion of the principal variable or $\dot{x}$ is generally taken of a constant but indeterminate magnitude, in the same manner as dx was assumed to be in the preceding pages; and if the value of $\frac{\dot{u}}{\dot{x}}$ be also variable, the rate of its variation compared with that of x, might in a similar manner be determined: this ratio would be represented by $\frac{\ddot{u}}{\dot{x}^2}$; and similarly, the expressions $\frac{\dddot{u}}{\dot{x}^3}, \frac{\ddddot{u}}{\dot{x}^4}$, &c. $\frac{u^{\dot{n}}}{\dot{x}^n}$ which are termed the Fluxional coefficients, correspond to $\frac{d^3u}{dx^3}, \frac{d^4u}{dx^4}$, &c. $\frac{d^nu}{dx^n}$ in the Differential Calculus, so that *Taylor's* Theorem is written

$$u' = u + \frac{\dot{u}}{\dot{x}}\frac{h}{1} + \frac{\ddot{u}}{\dot{x}^2}\frac{h^2}{1.2} + \frac{\dddot{u}}{\dot{x}^3}\frac{h^3}{1.2.3} + \&c.$$

90. From the short sketch of the principles of Fluxions just given, it is manifest that they are very easily reducible to the method of Limits, and that the rules and formulæ may be investigated and applied precisely in the same manner as those of the Differential Calculus have been.

It is objected to this mode of treating the subject, that recourse is had to the use of *Mechanical* considerations and terms in quantities which are entirely *Analytical*; but by the immediate transition proved in article (88) from this method to that of Limits which the Inventor of Fluxions has so copiously applied in his Principia, it cannot fail to be apparent, that had the objects of his enquiries been of an abstract analytical character and not mechanical or philosophical, the modifications of velocity and time, which he has introduced, would have been omitted, without in any degree affecting the nature and principles of the Calculus, as is sufficiently evident from the mode he has adopted in some of the Lemmas of the first section of the said work.

With respect to its notation, which is in many cases precarious, inconvenient and defective, the Fluxional Calculus is certainly inferior to the Differential.

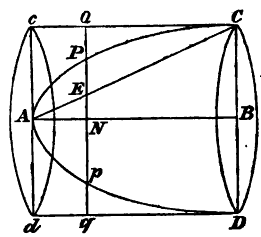
II. INDIVISIBLES.

91. In the method of *Indivisibles* introduced by *Bonaventura Cavalieri* commonly called *Cavalerius*, an Italian Mathematician who flourished near the beginning of the seventeenth century, all quantities are supposed to be made up of equal indivisible elementary portions: a line is considered to be made up of certain equal elements whose number is proportional to its length, a surface of a number of such lines, and a solid of a number of such surfaces: from which it follows that the magnitudes of all lines, surfaces and solids will be proportional to the number of elementary portions they re-

spectively contain: and the only difficulty in comparing magnitudes by this method is therefore the division of them into those elements, and in finding the relative numbers into which they can be divided.

This principle will be sufficiently illustrated by the following example, in which the content of a conical paraboloid is compared with that of its circumscribed cylinder.

Ex. Let CAD be a paraboloid, and $CcdD$ its circumscribed cylinder, having the same axis and altitude AB, and let



them both be cut by a plane perpendicular to the common axis, and passing through Q, P and N; then it is manifest, that the sections of the paraboloid and cylinder will be circles whose radii are NP and NQ respectively:

$$
\begin{aligned}
\text{now, } \odot \text{ rad} = NP : \odot \text{ rad} = NQ &:: \pi NP^2 : \pi NQ^2 \\
&:: NP^2 : NQ^2 \\
&:: NP^2 : BC^2 \\
&:: AN : AB, \text{ by the parabola,} \\
&:: NE : BC, \text{ by similar triangles,} \\
&:: NE : NQ:
\end{aligned}
$$

and the same may be proved of all other sections made in the same manner; $\therefore$ the sum of all the circles whose radii are NP : the sum of all the circles whose radii are NQ :: the sum of all the lines NE : the sum of all the lines NQ; but, by the principles of the method, the sum of all the circles whose radii are NP = the paraboloid CAD; the sum of all the circles whose radii are NQ = the cylinder $CcdD$; the sum of all

the lines NE = the triangle ABC; and the sum of all the lines NQ = the parallelogram AC;

therefore the paraboloid : its circumscribed cylinder

$$:: \text{triangle } ABC : \text{parallelogram } AC :: 1 : 2,$$

$$\text{and } \therefore \text{ the paraboloid } = \tfrac{1}{2} \text{ its circumscribed cylinder}$$
$$= \tfrac{1}{2}\pi BC^2 . AB.$$

It does not follow however in the same manner, that the convex surface of the paraboloid : the convex surface of the cylinder :: the sum of the circumferences of all the circles whose radii are NP : the sum of the circumferences of all the circles whose radii are NQ; because the perpendicular sections which divide the convex surface of the cylinder into parts of the same thickness, divide that of the paraboloid into portions whose relative thicknesses depend upon their positions; and hence it is evident, even from this instance, that great caution is requisite in the application of the method.

III. INFINITESIMALS.

92. The method of *Infinitesimals* adopted by LEIBNITZ as the foundation of his Differential Calculus, differs from the preceding in supposing all magnitudes to be composed of indefinitely small portions, admitting of all possible degrees of relative magnitude: such portions as are indefinitely small compared to any finite magnitude, are styled infinitesimals of the first order; such as are indefinitely small compared to these are called infinitesimals of the second order, and so on; and thence it follows, that infinitesimals of the n^{th} order are indefinitely less than those of the $(n-1)^{\text{th}}$ and indefinitely greater than those of the $(n+1)^{\text{th}}$ order: hence also, if h be an infinitesimal of the first order, since

$$\frac{h}{1} = \frac{h^2}{h} = \frac{h^3}{h^2} = \&\text{c.} = \frac{h^n}{h^{n-1}},$$

it is manifest that h^2, h^3, &c. h^n will be infinitesimals of the 2^{nd}, 3^{rd}, &c. n^{th} orders.

In this point of view, areas are supposed to be made up of parallelograms having one side finite, and another indefinitely small; solids of parallelopipeds of finite bases but indefinitely small altitudes, &c. and they are compared with each other by finding the sums of such indefinitely small rectangles, parallelopipeds, &c. by methods which the Calculus itself suggests.

Ex. 1. Let $u = ax^m$, and suppose u and x to receive the infinitesimal increments denoted by du and dx; then

$$u + du = a(x + dx)^m,$$

$$\therefore du = a(x+dx)^m - ax^m = max^{m-1}dx + \frac{m(m-1)}{1.2}ax^{m-2}dx^2$$

$$+\frac{m(m-1)(m-2)}{1.2.3}ax^{m-3}dx^3 + \&c.$$

now, since dx is an infinitesimal of the first order, it follows that dx^2, dx^3, &c. dx^n are infinitesimals of the 2^{nd}, 3^{rd}, &c. n^{th} orders respectively, and consequently that their sum may be rejected as an infinitesimal of an order superior to that of dx; whence we have

$$du = max^{m-1}dx, \text{ and } \therefore \frac{du}{dx} = max^{m-1}.$$

Ex. 2. Let $u = \tan x$, then on the supposition above made,

$$du = \tan(x+dx) - \tan x = \frac{\tan x + \tan dx}{1 - \tan x \tan dx} - \tan x$$

$$= \frac{\tan x + \tan dx - \tan x + (\tan x)^2 \tan dx}{1 - \tan x \tan dx}$$

$$= \frac{\{1 + (\tan x)^2\}\tan dx}{1 - \tan x \tan dx}:$$

but $\tan dx = dx$ is an infinitesimal of the first order, and therefore $\tan x \tan dx$ is indefinitely less than 1, whence

$$du = \{1 + (\tan x)^2\}dx = (\sec x)^2 dx, \text{ and } \therefore \frac{du}{dx} = (\sec x)^2.$$

Ex. 3. Let $u=xy$, then as before,

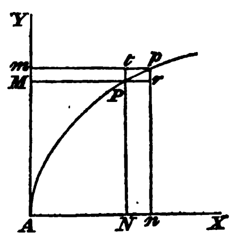
$$du=(x+dx)(y+dy)-xy=ydx+xdy+dxdy;$$

but since $1 : dx :: dy : dxdy$,

it follows that $dxdy$ is an infinitesimal of the second order; whence there remains only

$$du=ydx+xdy, \text{ and } \therefore \frac{du}{dx}=y+x\frac{dy}{dx}.$$

The result of this example may easily be exhibited geometrically, for if



we suppose $AN=x$, $NP=AM=y$, $Nn=dx$ and $Mm=dy$, it is manifest that $Pn=ydx$, $Pm=xdy$, and $Pp=dxdy$, of which Pp having no finite dimension may be neglected when compared to the two others which have each one.

Ex. 4. Retaining the figure and notation of the last example, if S be the area of the mixtilinear figure ANP, since the arc and its chord are ultimately coincident, we have

$$dS=\square Pn+\Delta Prp=PN.Nn+\tfrac{1}{2}Pr.rp=ydx+\tfrac{1}{2}dxdy,$$

and the latter part being an infinitesimal of the second order, and therefore indefinitely less than the former, there remains

$$dS=ydx:$$

also, if s represent the arc AP, we get

$$ds=Pp=\sqrt{Pr^2+rp^2}=\sqrt{dx^2+dy^2}.$$

The figures Pn and Ppr are called respectively an infinitesimal, and sometimes an elementary, parallelogram and triangle.

93. The method of reasoning used in these examples is applicable to all other functions; and by a repetition of similar operations, if we have $u = fx$, we shall arrive successively at $du = f_1 x dx$, $d^2u = f_2 x dx^2$, $d^3u = f_3 x dx^3$, &c. $d^n u = f_n x dx^n$, which are infinitesimals of the 1st, 2nd, 3rd, &c. nth orders respectively, and in form are precisely the same as have been obtained by the rules previously given.

Indeed the method of Infinitesimals manifestly amounts to the same thing as that of Limits, since the first term in any series involving ascending positive powers of dx, being indefinitely greater than the sum of all the rest, becomes the limit of the sum.

Also, the coefficients

$$\frac{du}{dx}, \frac{d^2u}{dx^2}, \frac{d^3u}{dx^3}, \text{ \&c. } \frac{d^nu}{dx^n},$$

being each expressed by the quotient of one infinitesimal divided by another of the same order, are finite and determinate functions of x: and hence as before, we obtain the general developement

$$u' = u + \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\frac{h^3}{1.2.3} + \text{\&c.}$$

94. Even from the few examples above given, it is evident that this method is exceedingly simple in its applications; but by the introduction of indefinitely small quantities of succeeding orders, of which we can have nothing more than a mathematical idea, the first principles become much less distinct and clear than either those of the method which has here been adopted, or of the method of Fluxions above explained, with the latter of which they have little or nothing in common, though they lead to the same results, and their Author for some time disputed with Newton the credit of the invention.

IV. DERIVED FUNCTIONS.

95. In all the views of the subject hitherto discussed there is involved a degree of metaphysical difficulty which the mind does not at once thoroughly comprehend, and they require to have introduced into them other considerations, besides those which ordinary analysis suggests, before the required conclusions can be obtained. *M. Lagrange*, who was duly sensible of these apparent difficulties and with a view of obviating them, presented his Calculus of *Derived Functions* upon grounds purely algebraical and entirely divested of all such perplexity whatever.

In this system, u being, as before, supposed to represent any function whatever of x, and u' the same function of $x+h$, it is demonstrated from the principles of common Algebra, that the relation subsisting among the quantities u, u', x and h is comprised in the equation

$$u' = u + \frac{1}{1} ph + \frac{1}{1.2} qh^2 + \frac{1}{1.2.3} rh^3 + \&c.$$

wherein p, q, r, &c. are derived from the original function, and from each other in succession, according to fixed laws, by means of general rules depending upon the nature of the function under consideration; and if, for the sake of uniformity of notation, dx be put in the place of h, or

$$u' = u + \frac{1}{1} p\,dx + \frac{1}{1.2} q\,dx^2 + \frac{1}{1.2.3} r\,dx^3 + \&c.$$

the quantities $p\,dx$, $q\,dx^2$, $r\,dx^3$, &c. are defined to be the first, second, third, &c. *Differentials* of u, and the coefficients p, q, r, &c. which are all functions of x, and here usually written $f'x$, $f''x$, $f'''x$, &c. are styled the first, second, third, &c. derived functions of the original.

It is manifest that this system assumes for its basis the expansion of *Taylor's* Theorem, which it establishes by means of common Algebra.

Ex. 1. Let $u = a x^m$, then by the binomial theorem,

$$u' = ax^m + max^{m-1}h + m(m-1)ax^{m-2}\frac{h^2}{1.2}$$

$$+ m(m-1)(m-2)ax^{m-3}\frac{h^3}{1.2.3} + \&c.,$$

or putting dx in the place of h, for the reason above assigned,

$$u' = ax^m + max^{m-1}dx + \frac{1}{1.2}m(m-1)ax^{m-2}dx^2$$

$$+ \frac{1}{1.2.3}m(m-1)(m-2)ax^{m-3}dx^3 + \&c.$$

now, according to the notation adopted before, we have

$u = ax^m$;

$du = max^{m-1}dx$, $\dfrac{du}{dx} = max^{m-1}$;

$d^2u = m(m-1)ax^{m-2}dx^2$, $\dfrac{d^2u}{dx^2} = m(m-1)ax^{m-2}$;

$d^3u = m(m-1)(m-2)ax^{m-3}dx^3$, $\dfrac{d^3u}{dx^3} = m(m-1)(m-2)ax^{m-3}$;

&c. = &c. ....

also, ax^m is the original function fx;

max^{m-1} is the first derived function $f'x$;

$m(m-1)ax^{m-2}$.........second..........$f''x$;

$m(m-1)(m-2)ax^{m-3}$...third...........$f'''x$;

&c.......................................

Ex. 2. Let $u = a^x$, then by the exponential theorem we have

$$u' = a^{x+h} = a^x a^h = a^x\left\{1 + \frac{kh}{1} + \frac{k^2h^2}{1.2} + \frac{k^3h^3}{1.2.3} + \&c.\right\}$$

$$=a^x+\frac{1}{1}ka^x dx+\frac{1}{1.2}k^2a^x dx^2+\frac{1}{1.2.3}k^3a^x dx^3+\&c.$$

by putting dx in the place of h: therefore, as before,

$$u=a^x;$$

$$du=ka^x dx,\quad \frac{du}{dx}=ka^x;$$

$$d^2u=k^2a^x dx^2,\quad \frac{d^2u}{dx^2}=k^2a^x;$$

$$d^3u=k^3a^x dx^3,\quad \frac{d^3u}{dx^3}=k^3a^x;$$

&c..............................

and a^x is the original function fx;

ka^x is the first derived function $f'x$;

k^2a^x ...... second .... $f''x$;

k^3a^x ...... third .... $f'''x$;

&c..............................

Ex. 3. Let $u=$ *Napierian* log x, then by algebraical principles we can prove

$$u'=\log(x+dx)=\log x+\frac{dx}{x}-\frac{dx^2}{2x^2}+\frac{dx^3}{3x^3}-\&c.$$

and $u=\log x$;

$$du=\frac{dx}{x},\quad \frac{du}{dx}=\frac{1}{x};$$

$$d^2u=-\frac{dx^2}{2x^2},\quad \frac{d^2u}{dx^2}=-\frac{1}{2x^2};$$

$$d^3u=\frac{dx^3}{3x^3},\quad \frac{d^3u}{dx^3}=\frac{1}{3x^3};$$

&c......................

where log x is the original function fx;

$\frac{1}{x}$ is the first derived function $f'x$;

$-\frac{1}{2x^2}$ ...... second .... $f''x$;

$\frac{1}{3x^3}$ ...... third .... $f'''x$;

&c............................

Ex. 4. Let $u = \sin x$, then by using the algebraical values of $\sin h$ and $\cos h$ found in plane Trigonometry, we have

$$u' = \sin(x+h) = \sin x \cos h + \cos x \sin h$$

$$= \sin x \left\{1 - \frac{h^2}{1.2} + \frac{h^4}{1.2.3.4} - \&c.\right\}$$

$$+ \cos x \left\{h - \frac{h^3}{1.2.3} + \frac{h^5}{1.2.3.4.5} - \&c.\right\}$$

$$= \sin x + \cos x\,dx - \frac{1}{1.2}\sin x\,dx^2 - \frac{1}{1.2.3}\cos x\,dx^3 + \&c.$$

and $u = \sin x$;

$$du = \cos x\,dx, \quad \frac{du}{dx} = \cos x;$$

$$d^2u = -\sin x\,dx^2, \quad \frac{d^2u}{dx^2} = -\sin x;$$

$$d^3u = -\cos x\,dx^3, \quad \frac{d^3u}{dx^3} = -\cos x;$$

&c......................................

where $\sin x$ is the original function fx;

$\cos x$ is the first derived function $f'x$;
$-\sin x$ ...... second ....$f''x$;
$-\cos x$ ...... third ....$f'''x$;
&c....................................

96. The instances just given have shewn us the possibility of deducing from the functions proposed, a series of new functions called Differential Coefficients, and we will now demonstrate how this Theory may be extended to any function whatever.

Assuming, as in (74), the indeterminate form,

$$u' = u + Ph^{\alpha} + Qh^{\beta} + Rh^{\gamma} + \&c.$$

and supposing that when h is changed into $h+k$, u' is changed into u'', we shall have

$$u'' = u + P(h+k)^{\alpha} + Q(h+k)^{\beta} + R(h+k)^{\gamma} + \&c. \qquad (1):$$

again, it is obvious that u' will be changed into u'' if x be changed into $x+k$; and on this hypothesis let

u become $u + Pk^{\alpha} + \&c.$
P become $P + P'k^{\alpha'} + \&c.$
Q become $Q + Q'k^{\alpha''} + \&c.$
R become $R + R'k^{\alpha'''} + \&c.$
&c.........................

whence by substitution, we immediately obtain

$$\left.\begin{aligned} u'' = u + Pk^{\alpha}_{i} &+ \&c. \\ + Ph^{\alpha} + P'h^{\alpha}k^{\alpha'} &+ \&c. \\ + Qh^{\beta} \quad + Q'h^{\beta}k^{\alpha''} &+ \&c. \\ + Rh^{\gamma} \quad + R'h^{\gamma}k^{\alpha'''} &+ \&c: \end{aligned}\right\} \quad (2)$$

making therefore $k = h$ in the equations marked (1) and (2), we shall have

$$u'' = u + P2^{\alpha}h^{\alpha} + Q2^{\beta}h^{\beta} + R2^{\gamma}h^{\gamma} + \&c.$$

and $u'' = u + Ph^{\alpha} + \&c.$

$$\begin{aligned} &+ Ph^{\alpha} + P'h^{\alpha+\alpha'} + \&c. \\ &\qquad + Qh^{\beta} \quad + Q'h^{\beta+\alpha''} + \&c. \\ &\qquad\qquad + Rh^{\gamma} \quad + R'h^{\gamma+\alpha'''} + \&c. \end{aligned}$$

whence, equating the corresponding terms in these two results, we have

$P2^{\alpha}h^{\alpha} = 2Ph^{\alpha}$, and $\therefore\ \alpha = 1$: similarly $\alpha' = \alpha'' = \alpha''' = \&c. = 1$:

and this gives

$$u' = u + Ph + Qh^{\beta} + Rh^{\gamma} + \&c.$$

$$= u + h\left\{P + Qh^{\beta-1} + Rh^{\gamma-1} + \&c.\right\};$$

but since P is the value of the expression between the brackets when $h = 0$, the same process which proved $\alpha = 1$, will also shew that $\beta - 1 = 1$, and $\therefore\ \beta = 2$: wherefore

$$u' = u + Ph + Qh^2 + Rh^{\gamma} + \&c.$$

$$= u + Ph + h^2\left\{Q + Rh^{\gamma-2} + \&c.\right\},$$

from which it may similarly be demonstrated that $\gamma - 2 = 1$, and $\therefore\ \gamma = 3$; and so on: and thus we have

$$u' = u + Ph + Qh^2 + Rh^3 + \&c.:$$

whence the expressions denominated (1) and (2), now become

$$u'' = u + P(h+k) + Q(h+k)^2 + R(h+k)^3 + \&c.$$

and $u'' = u + Pk + \&c.$

$$\begin{aligned} &+ Ph + P'hk + \&c. \\ &\qquad + Qh^2 \quad + Q'h^2k + \&c. \\ &\qquad\qquad + Rh^3 \quad + R'h^3k + \&c. \end{aligned}$$

$\therefore$ equating the coefficients of corresponding terms, we get

$$Q = \tfrac{1}{2}P', \quad R = \tfrac{1}{3}Q', \quad \&c. = \&c.$$

but by the definition it is assumed that

$$du = Ph = Pdx; \ \therefore P = \frac{du}{dx}:$$

$$\text{whence } P' = \frac{dP}{dx} = \frac{d^2u}{dx^2},$$

$$\text{and therefore } Q = \frac{1}{1.2} \frac{d^2u}{dx^2}:$$

$$\text{similarly, } Q' = \frac{dQ}{dx} = \frac{1}{1.2} \cdot \frac{d^3u}{dx^3},$$

$$\text{and therefore } R = \frac{1}{1.2.3} \frac{d^3u}{dx^3}; \text{ and so on:}$$

wherefore, we have at length obtained

$$u' = u + \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\frac{h^3}{1.2.3} + \&c.;$$

and consequently, if the developement of a proposed function of $x+h$, can by any means be expressed in a series ascending by integral and positive powers of h, the coefficients of

$$\frac{h}{1}, \ \frac{h^2}{1.2}, \ \frac{h^3}{1.2.3}, \ \&c. \ \frac{h^n}{1.2.3.\&c.\, n},$$

will be so many *Derived Functions* equivalent to the corresponding *Differential Coefficients*, denoted by

$$\frac{du}{dx}, \ \frac{d^2u}{dx^2}, \ \frac{d^3u}{dx^3}, \ \&c. \ \frac{d^nu}{dx^n}.$$

Ex. Let $u = yz$, where y and z are both functions of the same principal variable x; then will

$$u' = \{y + ph + \frac{1}{1.2}qh^2 + \&c.\} \, \{z + p'h + \frac{1}{1.2}q'h^2 + \&c.\}$$

$$= zy + (zp + yp')\frac{h}{1} + (zq + 2pp' + yq')\frac{h^2}{1.2} + \&c.$$

whence we have the first derived function

$$= zp + yp';$$

$$\text{but } p = \frac{dy}{dx} \text{ and } p' = \frac{dz}{dx}, \text{ by definition};$$

$$\therefore \frac{du}{dx} = \frac{zdy}{dx} + \frac{ydz}{dx}, \text{ and } du = zdy + ydz:$$

again, the second derived function will be

$$zq + 2pp' + yq' = z\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\frac{dz}{dx} + y\frac{d^2z}{dx^2}:$$

$$\therefore \frac{d^2u}{dx^2} = \frac{zd^2y + 2dydz + yd^2z}{dx^2},$$

and $d^2u = zd^2y + 2dydz + yd^2z$; and so on.

Similar modes of proceeding may be adopted for functions of other forms.

97. This method, like those which precede it, is easily reducible to the method of Limits; for since

$$u' - u = f'x\frac{h}{1} + f''x\frac{h^2}{1.2} + f'''x\frac{h^3}{1.2.3} + \&c.$$

we have

$$\frac{u' - u}{h} = f'x + f''x\frac{h}{1.2} + f'''x\frac{h^2}{1.2.3} + \&c.$$

the limit of which being taken, we manifestly have $\dfrac{du}{dx} = f'x$;

and we observe that the results of all the examples are precisely the same as have been found by the other methods in the preceding pages.

V. RESIDUAL ANALYSIS.

98. The notion of establishing this kind of Calculus upon principles purely algebraical, seems however to have originated with *Mr.* JOHN LANDEN a celebrated English Mathematician who flourished about the middle of the 18th century. In what is termed his ***Residual Analysis***, the first object is to exhibit the algebraical developement of the difference of the same functions of the quantities x and x' divided by the difference of the quantities themselves, or the developement of the expression

$$\frac{f(x') - f(x)}{x' - x}:$$

and afterwards to find what is called the *Special Value* of the result when x' is made $= x$ and when therefore all trace of the divisor $x' - x$ has disappeared: this has in fact been done in the examples of article (18), and the possibility of obtaining *generally* this special value depends entirely upon that of the developement proved in (96),

$$u' = u + Ph + Qh^2 + Rh^3 + \&c.$$

Hence if

$$x' = x + h \text{ and } f(x') = f(x) + Ph + Qh^2 + Rh^3 + \&c.$$

we shall have

$$\frac{f(x') - f(x)}{x' - x} = \frac{Ph + Qh^2 + Rh^3 + \&c.}{h}$$

$$= P + Qh + Rh^2 + \&c.:$$

and this, on the supposition that $h = 0$ or $x' = x$, becomes $= P$: in other words, the *Special Value* which it is the object of *Landen's* Calculus to ascertain, is identical with the *Fluxional* and *Differential Coefficient* above considered.

99. In point of metaphysical simplicity, *Lagrange's* system has undoubtedly great advantages over all the others; but it may be remarked, that it implies a previous knowledge

of the methods of developing all kinds of functions of $x+h$ in integral ascending positive powers of h, which in many cases cannot be effected without great prolixity of operation. This difficulty of effecting the developement in such terms has generally been found in practice by the younger students greatly to exceed that met with in determining the value of a ratio whose terms are evanescent; in addition to which, *Maclaurin's* and *Taylor's* Theorems, when once established by means of the method of limits, enable us, as has been already seen, to determine, with great ease, the developements of many functions, whose expansion by common Algebra would be exceedingly tedious. On these accounts it is that a preference has been given to the method of Limits adopted as the basis of the present performance.

To the mathematical student when further advanced, the short and imperfect sketches just given of the different views which have been taken of the subject, will naturally suggest the necessity of having recourse to the original works of their respective authors, from which all here said respecting them has been extracted, but slightly modified and altered with the view of being better adapted to the reading of junior mathematicians.

CHAP. VI.

On the Values of Functions involving Multipliers and Divisors which in particular cases become evanescent or infinite.

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100. The value of a ratio, or of a fraction which is equivalent to it, as has been before observed, does not depend upon the absolute magnitudes of its terms; and it has been seen in some of the preceding pages, that the real value of a fraction whose numerator and denominator are evanescent, or of what is usually termed a *Vanishing Fraction*, may be obtained by dividing each of them by the factor or factors which cause them to assume this particular form.

This is exemplified in the Geometrical Series

$$1 + x^2 + x^4 + \text{\&c. continued to } n \text{ terms},$$

whose sum according to the common rule, would be $\frac{x^{2n}-1}{x^2-1}$; but in the particular case of $x=1$, this becomes $\frac{0}{0}$, which affords no means of obtaining the real value.

If however, we divide both the numerator and denominator of the fraction $\frac{x^{2n}-1}{x^2-1}$ by $x-1$, we have the sum of the series

$$= \frac{x^{2n-1} + x^{2n-2} + \text{\&c. to } 2n \text{ terms}}{x+1},$$

which, when x is made $= 1$, becomes

$$= \frac{1 + 1 + \text{\&c. to } 2n \text{ terms}}{1+1} = \frac{2n}{2} = n.$$

Further, let $u = \frac{P}{Q}$, where P and Q are respectively equal to $X(x-a)^m$ and $X'(x-a)^{m'}$; then it is evident that if x take the particular value a, we shall have corresponding thereto,

$$u = \frac{X(a-a)^m}{X'(a-a)^{m'}} = \frac{0}{0}:$$

but though it assume this *indeterminate* form, the value of the fraction may be an evanescent, finite or infinite quantity, depending upon the relative values of the indices m and m': for

if m be greater than m', then $u = \dfrac{X(a-a)^{m-m'}}{X'} = 0$;

if m be equal to m', then $u = \dfrac{X(a-a)^{m-m}}{X'} = \dfrac{X}{X'}$;

if m be less than m', then $u = \dfrac{X}{X'(a-a)^{m'-m}} = \infty$:

the functions X and X' being supposed now to involve a in the place of x, and not to contain any factor, which, on this hypothesis, renders them either evanescent or infinite.

Whenever, therefore, an expression presents itself under the form $\frac{0}{0}$, we are not to conclude that its actual value $= 0$; but it must manifestly be our object to disengage from the numerator and denominator, those factors which in the particular case become evanescent.

101. Cor. 1. If when a particular value is assigned to the principal variable, the function $u = \frac{P}{Q}$ become of the form $\frac{\infty}{\infty}$, we may without altering the value of the fraction, divide both the numerator and denominator of it by PQ, so that

$$u=\frac{P}{Q}=\frac{\left(\frac{P}{PQ}\right)}{\left(\frac{Q}{PQ}\right)}=\frac{\left(\frac{1}{Q}\right)}{\left(\frac{1}{P}\right)},$$

which in the particular case proposed will become

$$u=\frac{\left(\frac{1}{\infty}\right)}{\left(\frac{1}{\infty}\right)}=\frac{0}{0},$$

and may therefore be evanescent, finite or infinite as before: also, on the same hypothesis, we shall have

$$u=P-Q=\infty-\infty=\frac{1}{0}-\frac{1}{0}=\frac{0-0}{0}=\frac{0}{0};$$

and in the same manner, if $u=PQ$ where P and Q are such functions of x that a particular value of it renders P evanescent and Q infinite, we shall have

$$u=PQ=\frac{\left(\frac{PQ}{Q}\right)}{\left(\frac{1}{Q}\right)}=\frac{P}{\left(\frac{1}{Q}\right)}=\frac{0}{\left(\frac{1}{\infty}\right)}=\frac{0}{0}, \text{ as before.}$$

102. Cor. 2. It follows therefore, that when functions in particular cases assume any of the forms

$$u=\frac{\infty}{\infty},\quad u=\infty-\infty,\quad \text{or } u=0\,.\,\infty,$$

they must first be reduced to fractions, whose numerators and denominators become each $=0$, or to the form $u=\frac{0}{0}$, and then be divested of the factors which are the cause of this peculiarity affecting them.

When these factors are explicitly exhibited, as in the latter example of the preceding article, we have merely to proceed by the method there pointed out; but as it seldom happens that this is the case, recourse must generally be had to other expedients which shall now be explained.

103. *To find the true values of Functions which for particular values of the principal variable assume the form* $\frac{0}{0}$.

Let $u = \frac{P}{Q}$, where P and Q are two functions of the same principal variable x, such that, if x have the particular value a, they both become $= 0$;

then retaining the notation of the last Chapter, we have by *Taylor's* Theorem,

$$u' = \frac{P + \frac{dP}{dx}\frac{h}{1} + \frac{d^2P}{dx^2}\frac{h^2}{1.2} + \&c. + \frac{d^nP}{dx^n}\frac{h^n}{1.2.3.\&c.\,n} + \&c.}{Q + \frac{dQ}{dx}\frac{h}{1} + \frac{d^2Q}{dx^2}\frac{h^2}{1.2} + \&c. + \frac{d^nQ}{dx^n}\frac{h^n}{1.2.3.\&c.\,n} + \&c.},$$

which is true, independently of the values assigned to x and h: now if $x = a$, P and Q become each $= 0$, and therefore in this case

$$u' = \frac{\frac{dP}{dx}\frac{h}{1} + \frac{d^2P}{dx^2}\frac{h^2}{1.2} + \&c. + \frac{d^nP}{dx^n}\frac{h^n}{1.2.3.\&c.\,n} + \&c.}{\frac{dQ}{dx}\frac{h}{1} + \frac{d^2Q}{dx^2}\frac{h^2}{1.2} + \&c. + \frac{d^nQ}{dx^n}\frac{h^n}{1.2.3.\&c.\,n} + \&c.}$$

$$= \frac{\frac{dP}{dx} + \frac{d^2P}{dx^2}\frac{h}{1.2} + \&c. + \frac{d^nP}{dx^n}\frac{h^{n-1}}{1.2.3.\&c.\,n} + \&c.}{\frac{dQ}{dx} + \frac{d^2Q}{dx^2}\frac{h}{1.2} + \&c. + \frac{d^nQ}{dx^n}\frac{h^{n-1}}{1.2.3.\&c.\,n} + \&c.},$$

wherein a is to be substituted in the place of x in the differential coefficients of P and Q: hence if h be made $=0$, and therefore u' become u, we have on the given hypothesis

$$u = \frac{\frac{dP}{dx}}{\frac{dQ}{dx}} = \frac{dP}{dQ}, \text{ for the particular value:}$$

but if $\frac{dP}{dQ} = \frac{0}{0}$, we shall have, from the same expression, after dividing by h,

$$u \frac{\frac{d^2P}{dx^2}}{\frac{d^2Q}{dx^2}} = \frac{d^2P}{d^2Q}, \text{ for the particular value:}$$

and generally, if $\frac{P}{Q} = \frac{dP}{dQ} = \frac{d^2P}{d^2Q} = \&\text{c.} = \frac{d^{n-1}P}{d^{n-1}Q} = \frac{0}{0}$, the actual value of the function for the proposed case will be obtained from the equation

$$u = \frac{\frac{d^nP}{dx^n}}{\frac{d^nQ}{dx^n}} = \frac{d^nP}{d^nQ},$$

which will be finite or not, according as the numerator and denominator in the particular case are both finite, or one finite and the other evanescent or infinite.

Ex. 1. Let $u = \frac{x^m - a^m}{x^n - a^n}$; then will $x^m - a^m = P$, and $x^n - a^n = Q$, and if x be assumed $=a$, u becomes of the form $\frac{0}{0}$; hence the actual value of the function in this case

$$= \frac{dP}{dQ} = \frac{d(x^m - a^m)}{d(x^n - a^n)} = \frac{mx^{m-1}}{nx^{n-1}} = \frac{ma^{m-1}}{na^{n-1}} = \frac{m}{n}a^{m-n}.$$

Here it is easily discovered that the factor, which is common to the numerator and denominator and causes their evanescence, is $x-a$; and hence, by common algebraical division, we have

$$u=\frac{x^{m-1}+ax^{m-2}+\text{\&c. to } m \text{ terms}}{x^{n-1}+ax^{n-2}+\text{\&c. to } n \text{ terms}}=\frac{ma^{m-1}}{na^{n-1}}=\frac{m}{n}a^{m-n},$$

when x is made equal to a.

Ex. 2. Let $u=\dfrac{mx^{m+1}-(m+1)x^m+1}{x^2-2x+1}$; then if x be supposed $=1$, this assumes the form $\dfrac{0}{0}$: hence the corresponding value of u will be

$$=\frac{dP}{dQ}=\frac{m(m+1)x^m-m(m+1)x^{m-1}}{2x-2}=\frac{0}{0}, \text{ when } x=1:$$

therefore on the same principles, the particular value of u

$$=\frac{d^2P}{d^2Q}=\frac{m^2(m+1)x^{m-1}-m(m^2-1)x^{m-2}}{2}$$

$$=\frac{m^2(m+1)-m(m^2-1)}{2}=\frac{m(m+1)}{2}\{m-m+1\}$$

$$=\frac{m(m+1)}{1.2}, \text{ when } x \text{ is supposed } =1.$$

The factor $(x-1)^2$ is contained in both the numerator and denominator of the fraction here proposed, though, from the manner in which the terms are combined, it is not explicitly apparent in the former: after one differentiation however of each we have

$$\frac{dP}{dQ}=\frac{m(m+1)x^{m-1}(x-1)}{2(x-1)},$$

which exhibits the common evanescing factor $x-1$, and hence, when this is discarded and x is made $=1$, we have the particular value of $u=\dfrac{m(m+1)}{1.2}$, as before.

Ex. 3. Let $u = \dfrac{a^x - b^x}{x}$, which, if x be supposed $=0$, becomes $\dfrac{0}{0}$:

$$\text{hence we have } \frac{dP}{dQ} = \frac{d\,(a^x - b^x)}{dx} = \log a a^x - \log b b^x$$

$$= \log a - \log b, \text{ if } x \text{ be made} = 0,$$

$$= \log \frac{a}{b}, \text{ the corresponding value of } u.$$

Here, though we do not, as the quantities stand, immediately discover the existence of a vanishing factor in the numerator, a little consideration will bring it to our view.

For, if $k = \log a$ and $k' = \log b$, we know from the principles of algebra that

$$a^x = 1 + kx + \frac{k^2 x^2}{1.2} + \frac{k^3 x^3}{1.2.3} + \&c.;\; b^x = 1 + k'x + \frac{k'^2 x^2}{1.2} + \frac{k'^3 x^3}{1.2.3} + \&c.;$$

$$\therefore\; u = \frac{(k - k')\,x + (k^2 - k'^2)\,\dfrac{x^2}{1.2} + (k^3 - k'^3)\,\dfrac{x^3}{1.2.3} + \&c.}{x},$$

which explicitly exhibits the evanescing factor x in both numerator and denominator; and by the rejection of this, the true value of u is found $= k - k' = \log a - \log b = \log \dfrac{a}{b}$.

Ex. 4. Let $u = \dfrac{x^4 - 2ax^3 + 2a^3x - a^4}{x^3 - ax^2 - a^2x + a^3}$, which, x being made $= a$, becomes $\dfrac{0}{0}$: hence in this instance

$$\frac{dP}{dQ} = \frac{d\,(x^4 - 2ax^3 + 2a^3x - a^4)}{d\,(x^3 - ax^2 - a^2x + a^3)} = \frac{4x^3 - 6ax^2 + 2a^3}{3x^2 - 2ax - a^2} = \frac{0}{0},$$

if $x = a$:

$$\text{also } \frac{d^2P}{d^2Q} = \frac{d\,(4x^3 - 6ax^2 + 2a^3)}{d\,(3x^2 - 2ax - a^2)} = \frac{12x^2 - 12ax}{6x - 2a} = \frac{0}{4a} = 0,$$

on the same supposition, which is therefore the required value of u.

In this example, the particular value of the function is evanescent, as will also appear by dividing both the numerator and denominator by the vanishing quadratic factor $(x-a)^2$; for then

$$u = \frac{x^2 - a^2}{x + a} = x - a = 0.$$

Ex. 5. Let $u = \dfrac{a^2 - x^2}{a^5 - a^4x - 2a^3x^2 + 2a^2x^3 + ax^4 - x^5} = \dfrac{0}{0}$, when $x = a$; therefore

$$\frac{dP}{dQ} = \frac{-2x}{-a^4 - 4a^3x + 6a^2x^2 + 4ax^3 - 5x^4} = \frac{2a}{0} = \infty \text{, the value}$$

of u, when the magnitude a is assigned to x.

Here it is not difficult to perceive that $a - x$ is the factor which causes u to assume the form $\dfrac{0}{0}$, and by expunging this, the true value of u, which is infinite, may be obtained.

Ex. 6. Let $u = \dfrac{\cos x - \sin x + 1}{\cos x + \sin x - 1}$, which $= \dfrac{0}{0}$, if $x = \frac{1}{2}\pi$;

$$\text{then } \frac{dP}{dQ} = \frac{d\,(\cos x - \sin x + 1)}{d\,(\cos x + \sin x - 1)} = \frac{-\sin x - \cos x}{-\sin x + \cos x},$$

which, by making $x = \frac{1}{2}\pi$, renders the particular value of $u = 1$.

By a little reduction, the factor, common to the numerator and denominator of this fraction, may be made to disappear; for, by plane Trigonometry,

$$u = \frac{(1 + \cos x) - \sin x}{\sin x - (1 - \cos x)} = \frac{2\,(\cos\frac{1}{2}x)^2 - 2\sin\frac{1}{2}x\cos\frac{1}{2}x}{2\sin\frac{1}{2}x\cos\frac{1}{2}x - 2\,(\sin\frac{1}{2}x)^2}$$

$$= \frac{\cos\frac{1}{2}x\,(\cos\frac{1}{2}x - \sin\frac{1}{2}x)}{\sin\frac{1}{2}x\,(\cos\frac{1}{2}x - \sin\frac{1}{2}x)} = \frac{\cos\frac{1}{2}x}{\sin\frac{1}{2}x} = \cot\tfrac{1}{2}x,$$

which, when $x = \frac{1}{2}\pi$, and therefore $\frac{1}{2}x = 45^0$, becomes $= 1$: and it is evident, since then $\cos\frac{1}{2}x = \cos 45^0 = \sin\frac{1}{2}x$, that we have $\cos\frac{1}{2}x - \sin\frac{1}{2}x$ for the vanishing factor.

Ex. 7. Let $u = \dfrac{\tan(2m-1)x}{\tan x}$, which, when $x = \frac{1}{2}\pi$, becomes of the form $\pm\frac{\infty}{\infty}$: then by reduction to a proper form, we have

$$u = \frac{\sin(2m-1)x\cos x}{\cos(2m-1)x\sin x} = \frac{0}{0};$$

$$\text{therefore } \frac{dP}{dQ} = \frac{d\{\sin(2m-1)x\cos x\}}{d\{\cos(2m-1)x\sin x\}}$$

$$= \frac{(2m-1)\cos(2m-1)x\cos x - \sin(2m-1)x\sin x}{-(2m-1)\sin(2m-1)x\sin x + \cos(2m-1)x\cos x};$$

$$\text{whence } u = \frac{1}{2m-1}, \text{ when } x \text{ is made } = \tfrac{1}{2}\pi.$$

Ex. 8. Let $u = \dfrac{1}{\log x} - \dfrac{x}{\log x}$, which, if $x = 1$, becomes

$$= \infty - \infty:$$

but by reduction, $u = \dfrac{1-x}{\log x} = \dfrac{0}{0}$, on the same supposition; therefore $\dfrac{dP}{dQ} = \dfrac{d(1-x)}{d\log x} = -x = -1 =$ the corresponding value of u.

Since $\log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \&c.$ it is manifest that $u = \dfrac{(1-x)}{(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \&c.}$

$$= -\frac{1}{1 - \frac{1}{2}(x-1) + \frac{1}{3}(x-1)^2 - \&c.},$$

which, when $x = 1$, becomes $= -1$, $1-x$ being the common factor which renders it of the form $\frac{0}{0}$.

Ex. 9. Let $u = (1-x) \tan \frac{1}{2}\pi x$, which is of the form $0 . \infty$, when x is made $= 1$;

therefore, reducing the expression to a proper form, we have

$$u = \frac{(1-x) \sin \frac{1}{2}\pi x}{\cos \frac{1}{2}\pi x} = \frac{0}{0}, \text{ if } x = 1:$$

hence, the particular value of $u = \dfrac{d\{(1-x) \sin \frac{1}{2}\pi x\}}{d \cos \frac{1}{2}\pi x}$

$$= \frac{-\sin \frac{1}{2}\pi x + (1-x) \cos \frac{1}{2}\pi x \, \frac{1}{2}\pi}{-\sin \frac{1}{2}\pi x \, \frac{1}{2}\pi} = \frac{-\sin \frac{1}{2}\pi}{-\sin \frac{1}{2}\pi \, \frac{1}{2}\pi} = \frac{2}{\pi},$$

by assigning to x the proposed value 1.

104. In article (103) the numerator and denominator of the function u' have been supposed capable of developement by means of *Taylor's* Theorem, for the particular value of the principal variable which occasions the singularity in its form; and therefore, whenever this theorem fails to effect these particular developements as in (77), recourse must be had to the common algebraical methods of obtaining their expansions.

Thus, if $u = \frac{P}{Q}$, where P and Q become each $= 0$, by assigning to x, of which they are functions, the particular value a, let $a+h$ be substituted for x, and suppose that, by the operations of Algebra, we have obtained

$$u' = \frac{Ah^{\alpha} + Bh^{\beta} + Ch^{\gamma} + \&c.}{A'h^{\alpha'} + B'h^{\beta'} + C'h^{\gamma'} + \&c.};$$

then if α be greater than α', by dividing both terms by $h^{\alpha'}$, we have

$$u' = \frac{Ah^{\alpha-\alpha'} + Bh^{\beta-\alpha'} + Ch^{\gamma-\alpha'} + \&c.}{A' + B'h^{\beta'-\alpha'} + C'h^{\gamma'-\alpha'} + \&c.};$$

whence, if $h = 0$, or $x = a$, we get the value of $u = \frac{A \cdot 0}{A'} = 0$:

again, if α be equal to α', we obtain by dividing by h^{α},

$$u' = \frac{A + Bh^{\beta-\alpha} + Ch^{\gamma-\alpha} + \&c.}{A' + B'h^{\beta'-\alpha} + C'h^{\gamma'-\alpha} + \&c.},$$

which, if $h = 0$, or $x = a$, gives the particular value of $u = \frac{A}{A'}$:

also, if α be less than α', by division by h^{α}, we obtain

$$u' = \frac{A + Bh^{\beta-\alpha} + Ch^{\gamma-\alpha} + \&c.}{A'h^{\alpha'-\alpha} + B'h^{\beta'-\alpha} + C'h^{\gamma'-\alpha} + \&c.},$$

which, on the supposition that $x = a$, or $h = 0$, gives the corresponding value of $u = \frac{A}{A' \cdot 0} = \infty$:

and hence the particular value of the function when reduced to a fractional form, will be evanescent, finite, or infinite, according as the least index of the developement of the numerator is greater than, equal to, or less than that of the denominator.

Ex. 1. Let $u = \frac{(x^3 - a^3)^{\frac{1}{2}}}{(x-a)^{\frac{1}{2}}}$, which becomes $\frac{0}{0}$, when $x = a$;

then putting $a + h$ for x, we have by common algebra

$$u' = \frac{\{(a+h)^3 - a^3\}^{\frac{1}{2}}}{(a+h-a)^{\frac{1}{2}}} = \frac{(3a^2h + 3ah^2 + h^3)^{\frac{1}{2}}}{h^{\frac{1}{2}}}$$

$$= \frac{h^{\frac{1}{2}}(3a^2 + 3ah + h^2)^{\frac{1}{2}}}{h^{\frac{1}{2}}} = (3a^2 + 3ah + h^2)^{\frac{1}{2}},$$

and if $h = 0$, or $x = a$, we get the particular value of

$$u=(3a^2)^{\frac{1}{2}}=\pm a\sqrt{3}.$$

Ex. 2. Let $u=\dfrac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}$;

therefore when $x=a$, $u=\dfrac{0}{0}$:

hence for x put $a+h$, and by the binomial Theorem, we shall have

$$\sqrt{x}-\sqrt{a}=\frac{1}{2}\,\frac{h}{\sqrt{a}}-\&c.,\quad \sqrt{x-a}=\sqrt{h},$$

and, $\sqrt{x^2-a^2}=\sqrt{2ah+h^2}=\sqrt{2ah}+\dfrac{h^2}{2\sqrt{2ah}}-\&c.$;

whence we obtain

$$u'=\frac{\dfrac{h}{2\sqrt{a}}-\&c.+\sqrt{h}}{\sqrt{2ah}+\dfrac{h^2}{2\sqrt{2ah}}-\&c.}=\frac{1+\dfrac{1}{2}\sqrt{\dfrac{h}{a}}-\&c.}{\sqrt{2a}+\dfrac{h}{2\sqrt{2a}}-\&c.};$$

and if h be supposed $=0$, or $x=a$, the particular value of

$$u=\frac{1}{\sqrt{2a}}:$$

and in this example, it is not difficult to discover that

$$\sqrt{\sqrt{x}-\sqrt{a}},$$

is the factor which produces the peculiarity in its form.

105. It may however be shewn independently of *Taylor's* Theorem, that the true value of a fraction, which, in a particular case, assumes the form $\dfrac{0}{0}$, may be obtained by the differentiation of the numerator and denominator, but that this method fails in the cases alluded to in the last article.

Since the fraction, when $x = a$, takes the form $\frac{0}{0}$, it is manifest that we may assume it to be of the form

$$u = \frac{X_1 (x - a)^m}{X'_1 (x - a)^{m'}},$$

in which X_1 and X_1' do not involve the factor $x - a$: hence, calling the numerator and denominator P and Q respectively, it is evident that $\frac{d^n P}{d^n Q}$ will be of the form

$$\frac{X_m (x-a)^m + X_{m-1} (x-a)^{m-1} + \&c. + m(m-1).\&c. X_1 (x-a)^{m-n}}{X'_{m'}(x-a)^{m'} + X'_{m'-1}(x-a)^{m'-1} + \&c. + m'(m'-1).\&c. X_1'(x-a)^{m'-n}}:$$

now, if m and m' be both whole numbers, and m greater than m', it is evident that $m - n$ will remain finite when $m' - n$ becomes $= 0$; and therefore when $x = a$, we have

$$u = \frac{0}{m'(m'-1).\&c. X'_1} = 0:$$

if m be equal to m', $m - n$ and $m' - n$ will become 0 at the same time; and thence, when $x = a$, we obtain

$$u = \frac{m(m-1).\&c. X_1}{m(m-1).\&c. X'_1} = \frac{X_1}{X'_1}:$$

if m be less than m', $m - n$ will become $= 0$, whilst $m' - n$ remains finite, and therefore if $x = a$, we find

$$u = \frac{m(m-1).\&c. X_1}{0} = \infty:$$

but, if m and m' be fractions, it is manifest that neither $m - n$ nor $m' - n$ can ever become $= 0$; and consequently that the vanishing factor $x - a$ cannot be disengaged from the numerator and denominator by the process of differentiation.

106. Although the method pointed out in article (103) fails to give the particular value of the function, when the index of the evanescing factor is fractional, it frequently happens that the one explained in article (104) may be used with advantage when the other does not fail, and differentiation would be tedious.

Ex. 1. Let $u = \dfrac{x^3 - 4ax^2 + 7a^2x - 2a^3 - 2a^2\sqrt{2ax - a^2}}{x^2 - 2ax - a^2 + 2a\sqrt{2ax - x^2}}$,

which becomes $= \dfrac{0}{0}$, when x is made $= a$:

suppose now $x = a + h$, then by substitution we shall have

$$u' = \frac{2a^3 + 2a^2h - ah^2 + h^3 - 2a^2\sqrt{a^2 + 2ah}}{-2a^2 + h^2 + 2a\sqrt{a^2 - h^2}};$$

$$\text{but } \sqrt{a^2 + 2ah} = a + h - \frac{h^2}{2a} + \frac{h^3}{2a^2} - \frac{5h^4}{8a^3} + \&c.,$$

$$\text{and } \sqrt{a^2 - h^2} = a - \frac{h^2}{2a} - \frac{h^4}{8a^3} - \&c.;$$

wherefore these series being substituted in the expression for u', and h being made $= 0$, the value required is $-5a$, which could not have been obtained by the other method, without four successive differentiations of both the numerator and denominator.

Ex. 2. Let $u = \dfrac{x^x - x}{\log x - x + 1}$, which becomes $\dfrac{0}{0}$, when $x = 1$;

then proceeding as before, we have

$$u' = \frac{(1+h)^{1+h} - (1+h)}{\log(1+h) - h} = \frac{h^2 + \frac{1}{1.2}(1+h)h^3 + \&c.}{-\frac{1}{2}h^2 + \frac{1}{3}h^3 - \&c.}$$

$$= \frac{1 + \frac{1}{1.2}(1+h)h + \&c.}{-\frac{1}{2} + \frac{1}{3}h - \&c.},$$

and if h be made $=0$, the corresponding value of $u = -2$.

This example would have required the numerator and denominator each to be differentiated twice before obtaining the real value of the fraction.

107. It sometimes happens that the differential coefficients of implicit functions, in consequence of assigning particular values to the independent variable, are made to assume the indeterminate form, and their true values must then be found upon the same principles. This will appear in the following examples.

Ex. 1. Let $(x^2+y^2)^2 = 2a^2(x^2-y^2)$, y being the dependent and x the independent variable; then, by the ordinary process, we find

$$\frac{dy}{dx} = \frac{x}{y}\left\{\frac{a^2 - x^2 - y^2}{a^2 + x^2 + y^2}\right\}:$$

now if $x = 0$ and $y = 0$, the former factor of this expression assumes the form $\frac{0}{0}$, whilst the latter becomes $= 1$; whence in this case we shall have

$$\frac{dy}{dx} = \frac{dx}{dy}, \text{ and } \therefore \frac{dy^2}{dx^2} = 1;$$

wherefore the corresponding values of $\frac{dy}{dx}$ are 1 and -1.

Ex. 2. Let there be given the equation

$$y^3 - 3axy + x^3 = 0;$$

$$\text{then will } \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax};$$

but since when x is made $=0$, y also becomes $=0$, we have

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax} = \frac{0}{0} = \frac{d(ay - x^2)}{d(y^2 - ax)} = \frac{a\frac{dy}{dx} - 2x}{2y\frac{dy}{dx} - a}:$$

and for the sake of conciseness putting $\frac{dy}{dx} = p$, we shall have

$$p = \frac{ap - 2x}{2py - a}, \text{ or } 2yp^2 - ap = ap - 2x;$$

$$\therefore p^2 - \frac{a}{y}p = -\frac{x}{y},$$

from which the values of p are found by the solution of a quadratic equation to be

$$\frac{a \pm \sqrt{a^2 - 4xy}}{2y} = \frac{a \pm (a - \frac{2xy}{a} - \frac{2x^2y^2}{a^3} - \&c.)}{2y};$$

and these expressions, when $x=0$ and $y=0$, become infinity and zero.

Ex. 3. Let the equation proposed be $ay^3 + x^4 = bx^2y$:

then $3ay^2dy + 4x^3dx = 2bxydx + bx^2dy$;

whence we find $\frac{dy}{dx} = \frac{2bxy - 4x^3}{3ay^2 - bx^2}$, which assumes the indeterminate form $\frac{0}{0}$ when $x=0$ and $y=0$:

hence, using a notation similar to that of the last example, we have

$$p = \frac{d(2bxy - 4x^3)}{d(3ay^2 - bx^2)} = \frac{bxp + by - 12x^2}{3ayp - bx}$$

$$= \frac{0}{0} = \frac{d(bxp + by - 12x^2)}{d(3ayp - bx)} = \frac{2bp}{3ap^2 - b},$$

when $x = 0$ and $y = 0$:

$$\therefore \text{ we have } p^3 - \frac{b}{a} p = 0, \text{ or } \left(p^2 - \frac{b}{a}\right) p = 0,$$

from which we obtain immediately,

$$p = 0, \text{ and } p^2 - \frac{b}{a} = 0, \text{ or } p = \pm \sqrt{\frac{b}{a}}:$$

in other words, the differential coefficient has three different values 0, $\sqrt{\frac{b}{a}}$ and $-\sqrt{\frac{b}{a}}$, corresponding to the value 0 of the independent variable.

108. Cor. The expressions $\frac{P}{Q}$, $\frac{dP}{dQ}$, $\frac{d^2P}{d^2Q}$, &c. cannot in any case assume the form $\frac{0}{0}$, *in infinitum*; for, since

$$u' = \frac{P + \frac{dP}{dx}\frac{h}{1} + \frac{d^2P}{dx^2}\frac{h^2}{1.2} + \&c.}{Q + \frac{dQ}{dx}\frac{h}{1} + \frac{d^2Q}{dx^2}\frac{h^2}{1.2} + \&c.},$$

if x were made $= a$, we should have the numerator and denominator each $= 0$ whatever be the value of h, which obviously cannot be the case.

These expressions may however assume the form $\frac{\infty}{\infty}$ in *infinitum*, whenever the particular value of the principal variable is such as to cause the failure of *Taylor's* Theorem, as may be seen in the examples of (103).

CHAP. VII.

On the greatest and least Values which Functions admit of, by assigning different Magnitudes to the principal Variable.

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109. Def. In the equation $u = fx$, or $f(u, x) = 0$, either of which expresses the relation subsisting between the function u and the principal variable x upon which it depends, if different values be given to x, it is evident that different values will *in general* be thus assigned to u; and if, by making x to pass successively through different degrees of magnitude, it be found that the corresponding values of u first increase and then decrease, it is evident that one of these values must be greater than either of those *immediately* preceding and following it: such a value of u is therefore called a *maximum*.

On the same hypothesis respecting x, if it appear that the values of u first decrease and afterwards increase, it follows in the same manner that one of them must be less than either of those *immediately* preceding and following it, and this value of u is styled a *minimum*.

Similarly, if, by continuing to assign other successive values to x, the same circumstances be observed to recur, the function is said to admit of so many *maxima* or *minima*, according as it ceases to increase and begins to decrease, or ceases to decrease and begins to increase.

In other cases when the function u increases or decreases continually, in consequence of the continual increase of x, it is manifest that its value admits neither of a *maximum* nor a *minimum* in the sense above explained, because each value is always either greater or less than that which *immediately* precedes it.

Since, therefore, according to the definitions just given, a function may admit of several maxima and minima, it follows that the distinguishing character of a *maximum* consists in its being *greater*, and that of a *minimum* in its being *less*, than either of the values *immediately* preceding and following it.

Ex. 1. Let there be proposed the equation

$$u = c \pm \sqrt{a^2 - (x - b)^2};$$

then if the upper sign be used, and x increase continually from 0 till it becomes $= b$, it is manifest that $(x - b)^2$ continually decreases till it becomes $= 0$, and consequently that the value of u increases till it becomes $= c + a$:

also, if x be still further increased, $(x - b)^2$ is likewise increased, and therefore the value of u will be diminished:

that is, u increases whilst x increases up to b, and afterwards decreases; or when $x = b$, the corresponding value of u is a *maximum* whose magnitude $= c + a$.

If the lower sign be used, it is evident that, whilst x increases up to b, the value of u continually decreases till it becomes $= c - a$, and afterwards increases by assigning to x successive values greater than b; that is, u decreases whilst x increases up to b, and afterwards increases, or u is a *minimum* whose magnitude is $c - a$, when x is made $= b$.

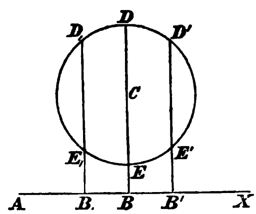
These results may be verified by substituting for x the values $b + h$ and $b - h$;

for, in the former case, the corresponding values of u are each $= c + \sqrt{a^2 - h^2}$, which is less than $c + a$; and in the latter, the values of $u = c - \sqrt{a^2 - h^2}$, which is greater than $c - a$.

These circumstances admit also of being exhibited geometrically; for, in fact, the equation

$$u = c \pm \sqrt{a^2 - (x - b)^2}$$

is that of a circle whose radius is a, and the co-ordinates of whose centre are b and c: hence



if $AB = b$ and $BC = c$, C will be the centre of the circle whose radius $CE = CD = a$: therefore

$$BD = BC + CD = c + a,$$

is evidently the greatest value of u,

$$\text{and } BE = BC - CE = c - a, \text{ is the least;}$$

also, if BB' and $BB_{,}$ be taken to represent h and $-h$ respectively, and the corresponding ordinates $B'\,E'\,D'$ and $B_{,}\,E_{,}\,D_{,}$ be drawn, it is clear that BD is greater than either $B'D'$ or $B_{,}D_{,}$, and BE less than either $B'E'$ or $B_{,}E_{,}$.

Ex. 2. Let $u = x^4 - 8ax^3 + 22a^2x^2 - 24a^3x + 12a^4$; then as x increases from 0, u decreases from $12a^4$, so that when x becomes $= a$, the corresponding value of $u = 3a^4$.

As x continues to increase from a, it also appears that the value of u continues to increase from $3a^4$, until x becomes $= 2a$, when the corresponding value of u becomes $4a^4$.

Afterwards, as x continues to increase from $2a$, the value of u decreases from $4a^4$, till x become $= 3a$, which renders the corresponding value of $u = 3a^4$.

After this, by continuing to assign to x successive values greater than $3a$, it is found that the values of u become continually greater and greater without ever again beginning to decrease.

Hence therefore it follows, that when the principal variable x has the values a and $3a$, the corresponding values of the function u are *minima*, whose magnitudes are each $= 3a^4$; and that when x becomes $= 2a$, the value of u corresponding to it is a *maximum* which $= 4a^4$.

In this example we may find, by substituting for x the values $4a$, $5a$, &c. that the corresponding values of u will be $12a^4$, $67a^4$, &c. which are greater than the *maximum* $4a^4$; but if h be a small quantity less than a, it will appear that of the pairs of quantities $4a+h$, $4a-h$; $5a+h$, $5a-h$, &c. one renders the corresponding value of u greater, and the other less than $12a^4$, $67a^4$, &c.; whereas $2a+h$ and $2a-h$ make the values of u corresponding to them *both* less than $4a^4$: this mode of reasoning proves the necessity of taking the values *immediately* preceding and following, in the definitions which have been given of a maximum and a minimum.

We might have arrived at the same conclusions by constructing the curve whose equation is

$$u = x^4 - 8ax^3 + 22a^2x^2 - 24a^3x + 12a^4;$$

which would shew that, according to the definition, the values a and $3a$ of the abscissa x render the ordinate u a minimum, and that the value $2a$ renders it a maximum.

Although similar expedients might be had recourse to in other cases, it is manifest that such a mode of proceeding would be both precarious and exceedingly tedious, particularly where surd quantities are concerned, and this the Differential Calculus furnishes the means of avoiding.

110. *To determine when a Function of one independent variable is a maximum or a minimum, and to investigate a criterion for distinguishing the one from the other.*

Let u representing any function of x, be the quantity whose maximum or minimum is required, and suppose u, u'

and $u_{\prime}$ to be the same functions of x, $x+h$ and $x-h$ respectively: then by *Taylor's* Theorem, we have

$$u' = f(x+h) = u + \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\frac{h^3}{1.2.3} + \&c.$$

and

$$u_{\prime} = f(x-h) = u - \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1.2} - \frac{d^3u}{dx^3}\frac{h^3}{1.2.3} + \&c.:$$

now, according as u is a maximum or a minimum, it is evident from what has been said, that u' and $u_{\prime}$ must be both less or both greater than u, however small h may be assumed; and we have seen in (76) that h can in general be assumed of such a magnitude that $\frac{du}{dx}h$ may be greater than the sum of all the terms which follow it;

whence, if $\frac{du}{dx}$ be either positive or negative, one of the quantities u', $u_{\prime}$ will be greater, and the other less, than u, which cannot be the case if u be either a maximum or a minimum;

and hence when u is either a maximum or minimum, it remains only that

$$\frac{du}{dx} = 0.$$

We have therefore, when the first differential coefficient vanishes,

$$u' = u + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\frac{h^3}{1.2.3} + \&c.$$

$$u_{\prime} = u + \frac{d^2u}{dx^2}\frac{h^2}{1.2} - \frac{d^3u}{dx^3}\frac{h^3}{1.2.3} + \&c.$$

which, on the principle that h can be assumed of such a magnitude that $\frac{d^2u}{dx^2}\frac{h^2}{1.2}$ may be greater than the sum of all the

terms which follow it, are both less than u when $\frac{d^2u}{dx^2}$ is negative, and both greater than u when $\frac{d^2u}{dx^2}$ is positive; that is, u will be a maximum or a minimum, according as the value of x, determined from the equation $\frac{du}{dx}=0$, renders the second differential coefficient $\frac{d^2u}{dx^2}$ negative or positive.

Ex. 1. Let it be required to find the value of x which renders the function $u = ax - x^2$, a maximum or a minimum.

Since $u = ax - x^2$, we have immediately by differentiation

$$\frac{du}{dx} = a - 2x, \text{ and } \frac{d^2u}{dx^2} = -2:$$

now because u is to be a maximum or a minimum, if $\frac{du}{dx}$ be put $= 0$, we shall have $a - 2x = 0$, and $\therefore x = \frac{1}{2}a$: and since the value of $\frac{d^2u}{dx^2}$ is negative, we conclude that $\frac{1}{2}a$ when substituted in the place of x renders the function $u = ax - x^2$ a *maximum*, whose value is $\frac{1}{4}a^2$.

If the function were $u = x^2 - ax$, we should have

$$\frac{du}{dx} = 2x - a = 0, \text{ or } x = \frac{1}{2}a;$$

and the value of $\frac{d^2u}{dx^2} = 2$, which is positive, proves that $\frac{1}{2}a$ renders the function $u = x^2 - ax$ a *minimum*, whose value is $-\frac{1}{4}a^2$.

Ex. 2. Let $u = x(a - x)^2$; then we have

$$\frac{du}{dx} = (a - 3x)(a - x), \text{ and } \frac{d^2u}{dx^2} = 6x - 4a:$$

but, if u be a maximum or a minimum, we must put

$\frac{du}{dx} = (a - 3x)(a - x) = 0$; and the equation $\frac{du}{dx} = 0$,

will be satisfied either by making $a - 3x = 0$, or $a - x = 0$;

therefore the roots of the equation $\frac{du}{dx} = 0$, are $\frac{1}{3}a$ and a:

now, if $\frac{1}{3}a$ be put for x, $\frac{d^2u}{dx^2} = 2a - 4a = -2a$,

and if a be put for x, $\frac{d^2u}{dx^2} = 6a - 4a = 2a$;

whence it follows that $\frac{1}{3}a$ and a, render u respectively a *maximum* and a *minimum*:

$$\text{also, the *maximum* value of } u = \frac{a}{3}\,\frac{4a^2}{9} = \frac{4a^3}{27},$$

and the *minimum* value of $u = a(a - a)^2 = a \cdot 0 = 0$.

Since $u = x^3 - 2ax^2 + a^2x$, it is manifest that if x be a *large* positive or negative quantity, the values of u will be greater or less than those already found, though they do not possess the character of a maximum or a minimum, as defined in (109).

Ex. 3. Let $u = x^4 - 8ax^3 + 22a^2x^2 - 24a^3x + 12a^4$; then

$$\frac{du}{dx} = 4x^3 - 24ax^2 + 44a^2x - 24a^3,$$

which $= 0$, when u is either a maximum or a minimum: whence we have

$$x^3 - 6ax^2 + 11a^2x - 6a^3 = 0,$$

$$\text{or } (x - a)(x - 2a)(x - 3a) = 0:$$

and the values of x which satisfy the equation $\frac{du}{dx} = 0$, are therefore a, $2a$ and $3a$:

$$\text{again,}\quad \frac{d^2u}{dx^2} = 12x^2 - 48ax + 44a^2,$$

in which if a, $2a$ and $3a$ be successively subtituted for x, the results will be

$$8a^2,\quad -4a^2 \text{ and } 8a^2;$$

whence each of the quantities a and $3a$ when substituted for x, renders the function a *minimum*, and $2a$ renders it a *maximum*: that is,

$$\begin{aligned} &\text{if } x = a, && u = 3a^4, && \textit{a minimum}: \\ &\text{if } x = 2a, && u = 4a^4, && \textit{a maximum}: \\ &\text{if } x = 3a, && u = 3a^4, && \textit{a minimum}. \end{aligned}$$

Ex. 4. Let $u = \dfrac{ax}{a^2 + x^2}$; then as before, we have

$$\frac{du}{dx} = \frac{a(a^2 - x^2)}{(a^2 + x^2)^2}, \text{ and } \frac{d^2u}{dx^2} = -\frac{2ax(3a^2 - x^2)}{(a^2 + x^2)^3}:$$

now, when u is a maximum or a minimum, we must make

$$\frac{du}{dx} = \frac{a(a^2 - x^2)}{(a^2 + x^2)^2} = 0:$$

whence we have $a(a^2 - x^2) = 0$, and $\therefore\ x = \pm a$:

$$\text{also, if } x = a,\quad \frac{d^2u}{dx^2} = -\frac{1}{2a^2};$$

$$\text{and if } x = -a,\quad \frac{d^2u}{dx^2} = \frac{1}{2a^2};$$

from which it follows that when $x = a$, the value of the function u is a *maximum;* and when $x = -a$, a *minimum:*

also, a and $-a$ being substituted for x give the *maximum* and *minimum* values of the function $= \frac{1}{2}$ and $-\frac{1}{2}$ respectively.

Ex. 5. Let $u = \sin x - \operatorname{vers} x$; then by differentiation,

$$\frac{du}{dx} = \cos x - \sin x, \text{ and } \frac{d^2u}{dx^2} = -\sin x - \cos x:$$

now, when u is a maximum or a minimum, we have, as before,

$$\frac{du}{dx} = \cos x - \sin x = 0;$$

$$\text{whence } \sin x = \cos x = \sin\left(\tfrac{1}{2}\pi - x\right) = \&c.$$

$$\text{and } \therefore\ x = \tfrac{1}{2}\pi - x = \&c., \text{ or } x = \tfrac{1}{4}\pi = 45^0, \text{ or } = \&c.:$$

also, for $x = 45^0$, $\dfrac{d^2u}{dx^2} = -\sin 45^0 - \cos 45^0 = -\dfrac{2}{\sqrt{2}} = -\sqrt{2}$,

which proves that the value of u thence resulting is a *maximum*.

It is evident that the number of values of x, which in this instance satisfy the equation $\dfrac{du}{dx} = 0$, is indefinitely great, and the value of u will be a maximum or a minimum, according as the value of $\sin x + \cos x$ is then positive or negative.

Ex. 6. Let $u^3 + x^3 - 3a^2x = 0$ be a proposed implicit function; then the operation of differentiation gives

$$3u^2\frac{du}{dx} + 3(x^2 - a^2) = 0:$$

$$\text{and } u^2\frac{d^2u}{dx^2} + 2u\frac{du^2}{dx^2} + 2x = 0:$$

but in case of a maximum or a minimum, we have $\dfrac{du}{dx} = 0$;

$$\therefore\ 3(x^2 - a^2) = 0, \text{ and } x = \pm a:$$

whence by substitution the proposed equation becomes

$$u^3 \mp 2a^3 = 0,$$

$$\text{and } \therefore\ u = \sqrt[3]{\pm 2a^3} = \pm a\sqrt[3]{2}:$$

also, in this case we have $\dfrac{d^2u}{dx^2} = -\dfrac{2x}{u^2} = \mp\dfrac{2}{a\sqrt[3]{4}}$,

whence the *maximum* and *minimum* values of u are $a\sqrt[3]{2}$ and $-a\sqrt[3]{2}$ respectively, correspondent to the values a and $-a$ of the independent variable.

Ex. 7. Let $u^2 - 2mxu + x^2 - a^2 = 0$; then, as before,

$$(u - mx)\frac{du}{dx} - (mu - x) = 0,$$

$$(u - mx)\frac{d^2u}{dx^2} - 2m\frac{du}{dx} + \frac{du^2}{dx^2} + 1 = 0:$$

from the former of which, since when u is a maximum or a minimum, $\frac{du}{dx} = 0$, we obtain $u = \frac{x}{m}$:

hence, by substitution in the proposed equation, we obtain

$$\frac{x^2}{m^2} - 2x^2 + x^2 - a^2 = 0, \text{ and } \therefore\ x = \pm\frac{ma}{\sqrt{1-m^2}};$$

$$\text{whence } u = \frac{x}{m} = \pm\frac{a}{\sqrt{1-m^2}}:$$

and to determine which of these values of u is a maximum and which a minimum, we have, when $x = \pm\frac{ma}{\sqrt{1-m^2}}$,

$$\frac{d^2u}{dx^2} = \frac{1}{mx - u} = \mp\frac{1}{a\sqrt{1-m^2}};$$

and this shews that u will be a *maximum* or a *minimum*, according as the value of x is $\frac{ma}{\sqrt{1-m^2}}$ or $\frac{-ma}{\sqrt{1-m^2}}$.

In the last two examples, in which u is considered as an *implicit* function of x, the same results would have been obtained, if u had been first rendered an *explicit* function of its principal variable.

111. Cor. If the characterising equation

$$\frac{du}{dx} = 0,$$

have the unequal possible roots a, b, c, &c.; and these be substituted successively in the equation

$$\frac{d^2u}{dx^2} = 0,$$

which is evidently the limiting equation of the former, the results will be alternately positive and negative; that is, the unequal roots of the equation

$$\frac{du}{dx} = 0,$$

taken in the order of their magnitudes render the function a minimum and a maximum alternately.

112. *To determine whether all the roots of the equation* $\frac{du}{dx} = 0$, *necessarily render the function* u *either a maximum or a minimum.*

The notation of the preceding articles being retained, if it appear that one or more roots of the equation

$$\frac{du}{dx} = 0,$$

satisfy also

$$\frac{d^2u}{dx^2} = 0, \text{ but not } \frac{d^3u}{dx^3} = 0,$$

we shall evidently have

$$u' = u + \frac{d^3u}{dx^3}\frac{h^3}{1.2.3} + \frac{d^4u}{dx^4}\frac{h^4}{1.2.3.4} + \&c.$$

$$u_{,} = u - \frac{d^3u}{dx^3}\frac{h^3}{1.2.3} + \frac{d^4u}{dx^4}\frac{h^4}{1.2.3.4} - \&c.;$$

and since h may be assumed of such a magnitude that

$$\frac{d^3u}{dx^3}\,\frac{h^3}{1.2.3}$$

shall be greater than the sum of all the terms which succeed it, it follows that one of the quantities u' and $u_{,}$ will be greater, and the other less, than u, if $\frac{d^3u}{dx^3}$ be any finite quantity, which is contrary to the nature of a maximum or a minimum: but if at the same time, the equation

$$\frac{d^3u}{dx^3} = 0,$$

be satisfied whilst the equation

$$\frac{d^4u}{dx^4} = 0,$$

is not fulfilled, we have again

$$u' = u + \frac{d^4u}{dx^4}\,\frac{h^4}{1.2.3.4} + \frac{d^5u}{dx^5}\,\frac{h^5}{1.2.3.4.5} + \&c.$$

$$u_{,} = u + \frac{d^4u}{dx^4}\,\frac{h^4}{1.2.3.4} - \frac{d^5u}{dx^5}\,\frac{h^5}{1.2.3.4.5} + \&c.,$$

which, when h is assumed of the proper degree of magnitude, will manifestly be both greater, or both less than u;

that is, u will be a *maximum* or a *minimum*, according as the value of x when substituted in $\frac{d^4u}{dx^4}$ gives a *negative* or a *positive* result.

Continuing to reason in the same manner, we conclude generally that a value of the principal variable can cause a function to be a maximum or a minimum, only when the differential coefficient, which it first renders finite, is of an even order; and that the function itself will be a maximum or a minimum, according as the value of that differential coefficient becomes then negative or positive.

Ex. 1. Let $u = b + (x-a)^3$; then we shall, as before, have

$$\frac{du}{dx} = 3(x-a)^2, \ \frac{d^2u}{dx^2} = 6(x-a), \text{ and } \frac{d^3u}{dx^3} = 6:$$

now, since from the nature of maxima and minima we must make

$$\frac{du}{dx} = 3(x-a)^2 = 0,$$

we obtain two values of x each $= a$; or in other words, if u admit of a maximum or a minimum, the corresponding value of x will be a;

but since this value of x renders $\dfrac{d^2u}{dx^2} = 0$, and $\dfrac{d^3u}{dx^3} = 6$,

it follows that b the corresponding value of u is neither a *maximum* nor a *minimum*.

Ex. 2. Let $u = x(a-x)^3$, be the proposed function: then

$$\frac{du}{dx} = (a-4x)(a-x)^2, \text{ and } \frac{d^2u}{dx^2} = 6(2x-a)(a-x):$$

hence the determining equation $\dfrac{du}{dx} = (a-4x)(a-x)^2 = 0$,

will be satisfied when $a - 4x = 0$, and when $(a-x)^2 = 0$;

that is, when $x = \frac{1}{4}a$ and $x = a$:

now, when $x = \frac{1}{4}a$, we have $\dfrac{d^2u}{dx^2} = -\dfrac{9a^2}{4}$,

which shews that $\frac{1}{4}a$ renders u a *maximum*, which $= \dfrac{27}{256}a^4$:

but when $x = a$, we have $\dfrac{d^2u}{dx^2} = 0$, and $\dfrac{d^3u}{dx^3} = -6a$;

therefore a being substituted in the place of x renders the function u which then $= 0$, neither a *maximum* nor a *minimum*.

Ex. 3. Let $u = x^x$; then, as before, we have

$$\frac{du}{dx} = x^x(1 + \log x), \text{ and } \frac{d^2u}{dx^2} = x^x\left\{\frac{1}{x} + (1 + \log x)^2\right\}:$$

$\therefore$ the characteristic equation is $\frac{du}{dx} = x^x(1 + \log x) = 0$,

which will be satisfied, both when

$$x^x = 0, \text{ and } 1 + \log x = 0:$$

from the latter of these we obtain $x = \frac{1}{e}$, which renders

$$\frac{d^2u}{dx^2} = \left(\frac{1}{e}\right)^{\frac{1}{e} - 1},$$

a positive quantity; and therefore, the corresponding value of

$$u = \left(\frac{1}{e}\right)^{\frac{1}{e}} = \frac{1}{\sqrt[e]{e}},$$

is a *minimum*.

If the equation $x^x = 0$, had been used, it is manifest that the corresponding values of $\frac{d^2u}{dx^2}$, $\frac{d^3u}{dx^3}$, &c. become $= 0$, and therefore the values of x which would be thus found, do not render the function either a *maximum* or a *minimum*.

Ex. 4. Let $u = e^x \cos 2x$; then we shall have

$$\frac{du}{dx} = e^x(\cos 2x - 2 \sin 2x), \quad \frac{d^2u}{dx^2} = -e^x(4 \sin 2x + 3 \cos 2x):$$

therefore from the equation of condition,

$$\frac{du}{dx} = e^x(\cos 2x - 2 \sin 2x) = 0,$$

we have $e^x = 0$, and $\cos 2x - 2 \sin 2x = 0$;

from the latter of which we obtain $\tan 2x = \frac{1}{2}$; and this gives

$$\sin 2x = \pm \frac{1}{\sqrt{5}} \text{ and } \cos 2x = \pm \frac{2}{\sqrt{5}};$$

wherefore if the upper sign be used, the corresponding value of u will evidently be a *maximum*, and if the lower, it will be a *minimum:*

it is also clear, that when $e^x = 0$, the values of $\frac{d^2u}{dx^2}$, $\frac{d^3u}{dx^3}$, &c. all vanish, or u is not by these means rendered either a *maximum* or a *minimum*.

113. Cor. 1. In some of the examples just given, it has appeared that the equation

$$\frac{du}{dx} = 0,$$

contains two equal roots which render the function neither a maximum nor a minimum; but it must be observed that this is not the case whatever may be the number of roots that are equal.

For, let the equation $\frac{du}{dx} = 0$, contain m equal roots, so that

$$\frac{du}{dx} = P(x-a)^m = 0;$$

then we have $\frac{d^2u}{dx^2} = (x-a)^m \frac{dP}{dx} + mP(x-a)^{m-1}$;

$$\frac{d^3u}{dx^3} = (x-a)^m \frac{d^2P}{dx^3} + 2m(x-a)^{m-1}\frac{dP}{dx} + m(m-1)P(x-a)^{m-2};$$

&c......................................

and generally

$$\frac{d^n u}{dx^n} = (x-a)^m X_m + (x-a)^{m-1} X_{m-1} + \&c.$$

$$+ m(m-1).\&c.\ P(x-a)^{m-n+1},$$

where X_m, X_{m-1}, &c.

represent the differential coefficients of P combined with constant quantities:

now, if m be *odd*, it is evident that the first differential coefficient of u which does not vanish, is of an even order; and therefore, a being substituted for x will render the function a maximum or a minimum, according as P then gives a negative or a positive result;

but if m be *even*, the first differential coefficient which does not vanish, is manifestly of an odd order, and indicates that the function does not, for this value of x, admit of either a maximum or a minimum.

Ex. 1. Let $u = a(x-b)^4$; then we have

$$\frac{du}{dx} = 4a(x-b)^3,\ \frac{d^2u}{dx^2} = 12a(x-b)^2,$$

$$\frac{d^3u}{dx^3} = 24a(x-b),\ \frac{d^4u}{dx^4} = 24a:$$

$$\text{now, } \frac{du}{dx} = 4a(x-b)^3 = 0,$$

contains three roots each equal to b;

but since the *fourth* differential coefficient becomes $24a$, it follows that b renders the function a *minimum* whose value is 0.

Ex. 2. Let $u = x^5 - 5ax^4 + 5a^2x^3 + a^5$; then, as before,

$$\frac{du}{dx} = 5x^4 - 20ax^3 + 15a^2x^2,$$

$$\frac{d^2u}{dx^2} = 20x^3 - 60ax^2 + 30a^2x,$$

$$\frac{d^3u}{dx^3} = 60x^2 - 120ax + 30a^2,$$

$$\frac{d^4u}{dx^4} = 120x - 120a, \quad \frac{d^5u}{dx^5} = 120:$$

but from $\frac{du}{dx} = 5x^4 - 20ax^3 + 15a^2x^2 = 0$, we have

$$x^2(x^2 - 4ax + 3a^2) = 0;$$

$$\therefore\ x^2 = 0, \text{ and } x^2 - 4ax + 3a^2 = 0,$$

the roots of which are 0, 0, a and $3a$:

if $x = 0$, we have $\frac{d^2u}{dx^2} = 0$ and $\frac{d^3u}{dx^3} = 30a^2$, therefore this value of the principal variable renders the function neither a *maximum* nor a *minimum:*

if $x = a$, then $\frac{d^2u}{dx^2} = -10a^3$, and if $x = 3a$, then $\frac{d^2u}{dx^2} = 90a^3$; which shew that a and $3a$ render the function a *maximum* and a *minimum*, whose respective values are $2a^5$ and $-26a^5$.

114. Cor. 2. If the equation $\frac{du}{dx} = 0$, contain several sets of equal roots, such that

$$\frac{du}{dx} = P(x-a)^m (x-b)^{m_1} (x-c)^{m_2} \&c. = 0;$$

the same mode of reasoning may be applied to shew that there will be one maximum or one minimum value of u belonging to each set when the indices m, m_1, m_2, &c. are odd, and none when they are even.

115. It sometimes happens that the equation

$$\frac{du}{dx} = 0,$$

contains impossible roots whilst its terms are all possible, and these roots may cause $\frac{d^2u}{dx^2}$ to be positive or negative without rendering the function a maximum or a minimum.

Ex. If we have $u = x^4 + 2x^2 + 3$; then will

$$\frac{du}{dx} = 4x^3 + 4x, \text{ and } \frac{d^2u}{dx^2} = 12x^2 + 4:$$

$$\text{now from the equation } \frac{du}{dx} = 0,$$

$$\text{we obtain } x = 0, \text{ and } x = \pm\sqrt{-1};$$

$$\text{also, if } x = 0, \frac{d^2u}{dx^2} = 4, \text{ and } u = 3, \text{ a } \textit{minimum}:$$

$$\text{if } x = \sqrt{-1}, \frac{d^2u}{dx^2} = -8, \text{ and } u = 2;$$

$$\text{if } x = -\sqrt{-1}, \frac{d^2u}{dx^2} = -8, \text{ and } u = 2;$$

and the last two values of u lie under the form of maxima; but they cannot be said to be either; because, in the investigation of the general principle, h was supposed to be assumed of such a magnitude that any one term of the values of u' and $u_{,}$ might be greater than the sum of all that follow it, which implies that all the differential coefficients are possible.

116. Should the terms of the developements of u' and $u_{,}$, to which recourse is had for the purpose of distinguishing a

maximum or a minimum, become infinite by substituting for x its value found from the equation

$$\frac{du}{dx} = 0,$$

it will be necessary to find their developements by other means.

Ex. 1. Let $u = b \pm (x - a)^{\frac{4}{3}}$; then we have

$$\frac{du}{dx} = \pm \frac{4}{3}(x - a)^{\frac{1}{3}}, \quad \frac{d^2u}{dx^2} = \pm \frac{4}{9(x - a)^{\frac{2}{3}}}, \text{ \&c.}:$$

$$\text{and } \therefore \text{ from } \frac{du}{dx} = 0, \text{ we have } x = a,$$

which manifestly renders $\frac{d^2u}{dx^2}$ as well as all the succeeding differential coefficients infinite, and therefore u' and $u_{,}$ do not for this value of x admit of being developed in ascending whole positive powers of h: but x being $= a$ renders

$$u' = b \pm h^{\frac{4}{3}} \text{ and } u_{,} = b \pm h^{\frac{4}{3}},$$

which are both greater or both less than u, and therefore b the value of u corresponding will be a *minimum* or a *maximum*, according as the upper or lower sign is used.

Ex. 2. Let $u = b \pm (x - a)^{\frac{5}{3}}$; then, as before,

$$\frac{du}{dx} = \pm \frac{5}{3}(x - a)^{\frac{2}{3}}, \quad \frac{d^2u}{dx^2} = \pm \frac{10}{9(x - a)^{\frac{1}{3}}};$$

and putting $\frac{du}{dx} = \pm \frac{5}{3}(x - a)^{\frac{2}{3}} = 0$, we get $x = a$, which renders all the succeeding differential coefficients infinite; but on this hypothesis, we have

$$u' = b \pm h^{\frac{5}{3}} \text{ and } u_{,} = b \mp h^{\frac{5}{3}},$$

which proves that the function u does not admit of either a *maximum* or a *minimum*

It scarcely need be observed that, when $\frac{d^2u}{dx^2}$ becomes impossible, and therefore cannot be considered either positive or negative, the function is neither a maximum nor a minimum.

117. Although it has been asserted that when $\frac{du}{dx} = 0$, the function u may be a maximum or a minimum, dependent upon the corresponding values of the succeeding differential coefficients, it is still possible that the particular value of the principal variable which renders the function a maximum or a minimum, may be the cause of the failure of *Taylor's* Theorem upon which the preceding articles of this Chapter have been made to depend.

Ex. Thus, if $u = b \pm (x - a)^{\frac{2}{3}}$, we shall have

$$\frac{du}{dx} = \pm \frac{2}{3(x-a)^{\frac{1}{3}}}, \quad \frac{d^2u}{dx^2} = \mp \frac{2}{9(x-a)^{\frac{4}{3}}}, \text{ \&c.}$$

which all become infinite when $x = a$, and render article (76) inapplicable in (110):

but if we substitute $a + h$ and $a - h$ in the place of x in the original function, we shall readily obtain

$$u' = b \pm h^{\frac{2}{3}} \text{ and } u_{,} = b \pm h^{\frac{2}{3}},$$

which are both greater or both less than the corresponding value of u, according as the upper or lower sign is made use of; and therefore if $x = a$, the function will be a *minimum* in the former case, and a *maximum* in the latter.

The same kind of reasoning will be applicable to the function

$$u = b \pm (x - a)^{\frac{m}{m+1}},$$

in which m is an even number; and it may be observed that when the function is of this particular form, the readiest way of obtaining the value of the principal variable which renders it a maximum or a minimum, is by putting

$$\frac{du}{dx} = \infty,$$

the values of x obtained from which will cause $\frac{d^2u}{dx^2}$ to be infinite and positive in the former case, and infinite and negative in the latter; and the whole may be verified by the algebraical substitution of $a + h$ and $a - h$ in the place of x.

118. From what has been said in the articles and examples defining and explaining a maximum and a minimum, it follows, as a matter of course, that when u is a maximum, $\frac{1}{u}$ is a minimum, and the contrary: these considerations however frequently lead us to maximum and minimum values of the function which the preceding views of the subject would not have enabled us to discover, though in many cases the additional values so determined are the results of expressions extended beyond what they are generally in ordinary enquiries designed to denote.

Ex. Let $u = \frac{(x + a)^3}{(x + b)^2} = \frac{1}{v}$; then, we shall have

$$\frac{du}{dx} = \frac{(x - 2a + 3b)(x + a)^2}{(x + b)^3},$$

$$\frac{d^2u}{dx^2} = 6(a - b)^2 \frac{x + a}{(x + b)^4};$$

$$\frac{dv}{dx} = \frac{(2a - 3b - x)(x + b)}{(x + a)^4},$$

$$\frac{d^2v}{dx^2} = \frac{2\{a^2 - 6ab + 6b^2 - (4a - 6b)x + x^2\}}{(x+a)^5};$$

now if u be be a maximum or a minimum, we have

$$\frac{du}{dx} = \frac{(x - 2a + 3b)(x + a)^2}{(x + b)^3} = 0,$$

which gives $x = 2a - 3b$ and $x = -a$:

also, if $x = 2a - 3b$, we have $\dfrac{d^2u}{dx^2} = \dfrac{9}{8(a-b)}$,

which indicates a *minimum* or a *maximum*, according as a is greater or less than b;

but if $x = -a$, then is $\dfrac{d^2u}{dx^2} = 0$,

or $-a$ renders the function neither a *maximum* nor a *minimum*.

Again, when v is a maximum or a minimum, we have

$$\frac{dv}{dx} = \frac{(2a - 3b - x)(x + b)}{(x + a)^4} = 0,$$

which will be satisfied when

$$x = 2a - 3b \text{ and } x = -b:$$

and if $x = 2a - 3b$, we have $\dfrac{d^2v}{dx^2} = -\dfrac{2}{81(a-b)^3}$,

which points out a *maximum* or a *minimum*, according as a is greater or less than b, agreeably to what is proved above;

but if $x = -b$, we get $\dfrac{d^2v}{dx^2} = \dfrac{2}{(a-b)^3}$,

and thence conclude that $-b$ renders v a *minimum* or a *maximum*, and therefore u a *maximum* or a *minimum*, according as a is greater or less than b.

In cases like the present, it is obvious that the results found above might have been obtained from the two equations

$$\frac{du}{dx} = 0 \text{ and } \frac{du}{dx} = \infty;$$

and by these means much trouble would have been avoided.

119. If the second or any of the succeeding differential coefficients assume the indeterminate form $\frac{0}{0}$, in consequence of assigning to x the particular values obtained from the equation

$$\frac{du}{dx} = 0,$$

their true values must be obtained by one of the methods pointed out in the last Chapter, and the conclusions must be drawn agreeably to the principles laid down in the preceding articles.

120. When any particular value of x renders u a maximum or a minimum, it is manifest that the corresponding value of $a+u$ will also be a maximum or a minimum, and that of $a-u$, a minimum or a maximum.

Similarly, if $v = \frac{au}{b}$ and $\therefore \frac{dv}{dx} = \frac{a}{b} \cdot \frac{du}{dx}$,

the same values of x which satisfy the equation $\frac{du}{dx} = 0$, will also satisfy the equation $\frac{dv}{dx} = 0$; that is, v is a maximum or a minimum at the same time as u.

Again, if $v = u^m$, we shall have $\frac{dv}{dx} = mu^{m-1}\frac{du}{dx}$;

therefore, when u is a maximum or a minimum, and conse-

quently $\frac{du}{dx} = 0$, it follows that $\frac{dv}{dx} = 0$, and therefore, that v is also a maximum or a minimum:

but since $\frac{dv}{dx}$ may become $= 0$, by the evanescence of mu^{m-1}, it is not necessary that all the values of x which satisfy the equation $\frac{dv}{dx} = 0$, should likewise fulfil the conditions of the equation $\frac{du}{dx} = 0$: or, in other words, values may be assigned to x, which will render v a maximum or a minimum, at the same time that u becomes the contrary or neither.

It is moreover possible that a value of x which makes $\frac{du}{dx} = 0$, may render $mu^{m-1} = \infty$, so that $\frac{dv}{dx}$ assumes the form $\frac{0}{0}$, which does not necessarily $= 0$; and in such a case, though u may be a maximum or a minimum, v may be neither.

Ex. Let $u = a^{\frac{4}{3}} - x^{\frac{1}{3}}(a - x)$; then if u be a maximum or a minimum, it is manifest that $x(a - x)^2$ will in general be a minimum or a maximum, and that the values of x, which answer the conditions of the former, will be included amongst those which belong to the latter:

$$\text{suppose } v = x(a - x)^2;$$

$$\text{then } \frac{dv}{dx} = (a - 3x)(a - x) \text{ and } \frac{d^2 v}{dx^2} = 6x - 4a:$$

$$\text{therefore from } \frac{dv}{dx} = 0, \text{ we get } x = \frac{a}{3} \text{ and } x = a;$$

if $x = \frac{a}{3}$, $\frac{d^2 v}{dx^2} = -2a$; $\therefore$ v is a *maximum*, and u a *minimum*:

if $x = a$, $\frac{d^2 v}{dx^2} = 2a$; $\therefore$ v is a *minimum*, and u a *maximum*:

again, we have

$$\frac{du}{dx} = -\frac{a}{2x^{\frac{3}{2}}} + \frac{3x^{\frac{1}{2}}}{2}, \text{ and } \frac{d^2u}{dx^2} = \frac{a}{4x^{\frac{3}{2}}} + \frac{3}{4x^{\frac{1}{2}}};$$

$$\text{whence from } \frac{du}{dx} = 0, \text{ we find } x = \frac{a}{3},$$

$$\text{which makes } \frac{d^2u}{dx^2} = \frac{3}{2}\sqrt{\frac{3}{a}},$$

and therefore u a *minimum*;

but the additional value a of x which renders v a *minimum*, makes $\frac{d^2u}{dx^2} = \frac{1}{\sqrt{a}}$, and indicates that the corresponding value of u would also be a *minimum*, of which the equation $\frac{du}{dx} = 0$, gives no account.

121. From the equations $v = a^u$, and $v = \log u$, we have

$$\frac{dv}{dx} = \log a \, a^u \frac{du}{dx} \text{ and } \frac{dv}{dx} = \frac{du}{u\,dx};$$

and therefore the values of x which satisfy the equation $\frac{du}{dx} = 0$, or render u a maximum or a minimum, will also fulfil the conditions of the equation $\frac{dv}{dx} = 0$, except the functions a^u and $\frac{1}{u}$, at the same time, become indefinitely great, in which case $\frac{dv}{dx}$ becomes of the form $\infty \, . \, 0$, or $\frac{0}{0}$, and may be evanescent, finite or infinite; and it is needless to observe that, except on the first supposition, a maximum or a minimum cannot thus be determined.

122. The preceding articles will, in many cases, greatly facilitate the discovery of ordinary maxima and minima, particularly when the functions are of a fractional, surd, or other complicated form.

Ex. 1. Let $u = \dfrac{x^{\frac{1}{2}}}{x+a}$; then taking logarithms, we have

$$v = \log\left\{\frac{x^{\frac{1}{2}}}{x+a}\right\} = \frac{1}{2}\log x - \log(x+a),$$

$$\text{and therefore } \frac{dv}{dx} = \frac{1}{2x} - \frac{1}{x+a} = 0;$$

whence $x = a$, which, as may be easily proved, renders the proposed function u a *maximum* whose value $= \dfrac{1}{2\sqrt{a}}$.

Ex. 2. Let $u = (\sin x)^m \{\sin(a-x)\}^n$; then will

$$v = \log u = m \log \sin x + n \log \sin(a-x), \text{ and}$$

$$\frac{dv}{dx} = m\frac{\cos x}{\sin x} - n\frac{\cos(a-x)}{\sin(a-x)} = 0;$$

whence $n \tan x = m \tan(a-x)$, from which x may be found by the solution of a quadratic:

$$\text{or, since } \tan x : \tan(a-x) :: m : n, \text{ we have}$$

$$\tan x + \tan(a-x) : \tan x - \tan(a-x) :: m+n : m-n,$$

$$\text{or } \sin a : \sin(2x-a) :: m+n : m-n;$$

$$\text{whence } \sin(2x-a) = \frac{m-n}{m+n}\sin a,$$

$$\text{or } 2x - a = \sin^{-1}\left(\frac{m-n}{m+n}\sin a\right);$$

$$\text{and therefore } x = \frac{a}{2} + \frac{1}{2}\sin^{-1}\left(\frac{m-n}{m+n}\sin a\right);$$

the value of u corresponding to which is easily proved to be a *maximum*.

123. In order to decide whether a value of the principal variable renders the function a maximum or a minimum, we have to ascertain whether this value makes the second differential coefficient negative or positive; and the determination of the value of this coefficient, which, by the ordinary method, might be laborious, will be much facilitated by means of the following considerations.

Suppose the first differential coefficient $\frac{du}{dx}$ to be of the form PQ, where P and Q are functions of x; then we have immediately

$$\frac{d^2u}{dx^2} = Q\frac{dP}{dx} + P\frac{dQ}{dx};$$

but since, $\frac{du}{dx} = 0$, we have $P = 0$ and $Q = 0$;

$\therefore$ when $P = 0$, the values of x give $\frac{d^2u}{dx^2} = Q\frac{dP}{dx}$,

and when $Q = 0$, the values of x give $\frac{d^2u}{dx^2} = P\frac{dQ}{dx}$,

in which we have merely to substitute in P and Q and their differential coefficients the values of x, and thus the magnitude and sign of $\frac{d^2u}{dx^2}$ will be determined.

Here it may be observed however that the values of x are not supposed to render either of the coefficients $\frac{dP}{dx}$ or $\frac{dQ}{dx}$ infinite.

Again, let $\frac{du}{dx}$ be of the form $\frac{P}{Q}$; then, as before, since

$$\frac{du}{dx} = 0,$$

$$\text{we have } \frac{d^2u}{dx^2} = \frac{1}{Q}\frac{dP}{dx} - \frac{P}{Q^2}\frac{dQ}{dx} = \frac{1}{Q}\frac{dP}{dx};$$

and thus the value of $\frac{d^2u}{dx^2}$ will be found as before, the same supposition being made respecting the magnitudes of the differential coefficients $\frac{dP}{dx}$ and $\frac{dQ}{dx}$.

Ex. 1. Let $u=(x-1)(2-x)^2$; then for a maximum or a minimum we have $\frac{du}{dx}=(2-x)(4-3x)=0$,

$$\text{whence } x=2, \text{ and } x=\frac{4}{3}:$$

$$\text{but } P=2-x,\ \frac{dP}{dx}=-1,\ Q=4-3x,\ \frac{dQ}{dx}=-3;$$

$$\text{therefore when } x=2,\ \frac{d^2u}{dx^2}=Q\frac{dP}{dx}=-(4-6)=2;$$

that is, 2 renders the function a *minimum* $=0$;

$$\text{and when } x=\frac{4}{3},\ \frac{d^2u}{dx^2}=P\frac{dQ}{dx}=-3\left(2-\frac{4}{3}\right)=-2;$$

or $\frac{4}{3}$ makes the corresponding value of the function a *maximum* $=\frac{4}{27}$.

Ex. 2. Let $u=\frac{x^2+3}{x+1}$; then $\frac{du}{dx}=\frac{x^2+2x-3}{(x+1)^2}=0$,

from which $x^2+2x-3=0$, and $\therefore x=1$ and $x=-3$:

now, $P=x^2+2x-3$, $\frac{dP}{dx}=2(x+1)$ and $Q=(x+1)^2$,

therefore when $x = 1$, $\frac{d^2u}{dx^2} = \frac{1}{Q}\frac{dP}{dx} = \frac{4}{4} = 1$, and the function u is a *minimum* whose value $= 2$:

also, when $x = -3$, $\frac{d^2u}{dx^2} = \frac{1}{Q}\frac{dP}{dx} = -\frac{4}{4} = -1$; that is, the corresponding value of u is a *maximum* $= -6$.

If we had made $x + 1 = \infty$, to satisfy the equation $\frac{du}{dx} = 0$, it is manifest that $\frac{d^2u}{dx^2}$ would have become $= 0$, which indicates neither a *maximum* nor a *minimum*.

Ex. 3. Let $au^3 - u^2x^2 + x^4 = 0$; then we have

$$(3\,au^2 - 2\,ux^2)\frac{du}{dx} - (2\,u^2x - 4\,x^3) = 0;$$

but when u is a maximum or a minimum, $\frac{du}{dx} = 0$, and therefore

$$2\,x\,(u^2 - 2\,x^2) = 0, \text{ whence } u^2 = 2\,x^2;$$

therefore by substitution we have

$$\pm\, 2\sqrt{2}\,ax^3 - 2\,x^4 + x^4 = 0,$$

which gives $x = \pm\, 2\sqrt{2}\,a$, and therefore $u = 4\,a$:

but since $\frac{du}{dx} = \frac{2\,u^2x - 4\,x^3}{3\,au^2 - 2\,ux^2} = \frac{P}{Q}$, we get the value of

$$\frac{d^2u}{dx^2} = \frac{1}{Q}\frac{dP}{dx}$$

$$= \frac{1}{3\,au^2 - 2\,ux^2}\left(4\,ux\frac{du}{dx} + 2\,u^2 - 12\,x^2\right)$$

$$= \frac{2u^2 - 12x^2}{3au^2 - 2ux^2}, \text{ since } \frac{du}{dx} = 0,$$

$$= \frac{1}{u}\left(\frac{2u^2 - 12x^2}{3au - 2x^2}\right) = \frac{1}{4a}\left(\frac{32a^2 - 96a^2}{12a^2 - 16a^2}\right) = \frac{4}{a},$$

from which it follows that when $x = \pm 2\sqrt{2}\,a$, the value of u is a *minimum* whose magnitude $= 4a$.

If we had taken $x = 0$ which also satisfies $\frac{du}{dx} = 0$, we should have found $u = 0$; but these values cause $3au^2 - 2ux^2$ to vanish, and do not satisfy the equation $\frac{du}{dx} = 0$, as appears from (107); and therefore we cannot conclude that the value of the function is either a maximum or a minimum.

124. Although the determining equation

$$\frac{du}{dx} = 0,$$

be not satisfied, and consequently u admit not of either a maximum or a minimum in the sense of the words as above explained; still, in some functions there may in this case be maxima and minima of another description, the existence of which is distinguished not by the relative magnitudes of preceding and suceeding values, because either of them may be impossible, but by those of two different preceding or two different succeeding values, corresponding to the same value of the principal variable.

Ex. 1. Let $u = ax^{\frac{3}{2}} \pm b(a - x)^{\frac{3}{2}}$; then if x be supposed $= a$, the two values of u coincide and become $= a^{\frac{5}{2}}$; also if x be greater than a, both values of u become imaginary, of whose magnitude we can therefore form no judgment: but if for x we substitute $a - h$, we shall have

$$u_{,} = a(a-h)^{\frac{3}{2}} \pm bh^{\frac{3}{2}} = a^{\frac{5}{2}} - \frac{3}{2}a^{\frac{3}{2}}h \pm bh^{\frac{3}{2}} + \&c.$$

which values are both manifestly less than $a^{\frac{5}{2}}$, when h is indefinitely diminished, and therefore, the double value of u, which $= a^{\frac{5}{2}}$, is in this case a *maximum*.

Ex. 2. Let us take $u^2 - 2ux - 2x^2 - 1 + 3x + x^3 = 0$, from which, by the nature of maxima and minima, we get

$$\frac{du}{dx} = \frac{2u - 3 + 4x - 3x^2}{2(u - x)} = 0,$$

$$\text{and thence } u = \frac{3}{2}x^2 - 2x + \frac{3}{2};$$

this, substituted in the proposed equation, gives

$$9x^4 - 32x^3 + 42x^2 - 24x + 5 = 0, \text{ or}$$

$$(x^2 - 2x + 1)(9x^2 - 14x + 5) = 0,$$

from which we obtain $x = 1$, $x = 1$; $x = 1$, and $x = \frac{5}{9}$: but when $x = 1$, we manifestly have $(u - 1)^2 = 0$, and $u = 1$;

also $\dfrac{du}{dx} = \dfrac{2u - 3 + 4x - 3x^2}{2(u - x)} = \dfrac{0}{0} = \dfrac{\dfrac{du}{dx} + 2 - 3x}{\dfrac{du}{dx} - 1}$, from (107);

and this gives the value of $\dfrac{du}{dx} = 1$, which indicates neither a *maximum* nor a *minimum* of the first kind:

also, when $x = \dfrac{5}{9}$, $u = \dfrac{23}{27}$, which is a *minimum* of the first kind, because the equation $\dfrac{du}{dx} = 0$ is in this case satisfied, and the second differential coefficient is then positive.

Again, since the proposed equation is equivalent to

$$(u - x)^2 - (1 - x)^3 = 0,$$

$$\text{we find } u = x \pm (1-x)^{\frac{3}{2}};$$

which, when $x = 1$, gives $u = 1$, a *maximum* of the second kind, because if $x = 1 - h$, the values of u are $1 - h \pm h^{\frac{3}{2}}$, which are both less than 1, when h is supposed very small.

Ex. 3. Let $u^3 - 3aux + x^3 = 0$; then, as before, we have

$$\frac{du}{dx} = \frac{au - x^2}{u^2 - ax} = 0; \text{ whence } au = x^2 \text{ and } u = \frac{x^2}{a};$$

therefore by substitution we obtain $x^6 - 2a^3x^3 = 0$,

$$\text{which gives } x = 0, \text{ and } x = a\sqrt[3]{2}:$$

the former of these renders

$$\frac{du}{dx} = \frac{au - x^2}{u^2 - ax} = \frac{0}{0} = \frac{a\frac{du}{dx} - 2x}{2u\frac{du}{dx} - a}, \text{ by (107)},$$

from which we may have $\frac{du}{dx} = 0$; but this does not point out that $u = 0$, is either a *maximum* or a *minimum* of the first kind, because $\frac{d^2u}{dx^2}$ cannot be shewn to be either positive or negative: and the latter makes $u = a\sqrt[4]{4}$, a *maximum*.

Again, let $x = 0 + h = h$, and $u = 0 + k = k$; therefore we have $k^3 - 3akh + h^3 = 0$, from which it follows that when these quantities are very small, k must be of one of the forms

$$k = \alpha\sqrt{h} \text{ and } k = \beta h^2:$$

$$\text{if } k = \alpha\sqrt{h}, \text{ we get } u' = \pm\sqrt{3ah}, \text{ when } x = h,$$

one of which is greater and the other less than 0; and therefore, $u = 0$, is not a *maximum* or a *minimum* of the second kind:

$$\text{but if } k = \beta h^2, \text{ we find } u' = \frac{h^2}{3a} = u_{,};$$

therefore both u' and $u_{,}$ are greater than 0, however small a positive or negative quantity h may be assumed, and consequently $u=0$, is a *minimum* of the first kind.

125. The same mode of reasoning may be pursued in the equation

$$u=fx\pm Fx(a-x)^{\frac{2m+1}{2n}},$$

wherein fx and Fx are any functions of x not involving the factor $a-x$, provided $2m+1$ be greater than $2n$;

and it may be observed that, when $x=a$, u which is then equal to the corresponding value of fx, will be a maximum or minimum of the second kind, according as $\dfrac{dfx}{dx}$ is positive or negative; for, the two values of u become equal to each other when $x=a$, and afterwards impossible:

but if $\dfrac{dfx}{dx}=0$, there will be neither a maximum nor a minimum, unless $\dfrac{2m+1}{2n}$ be greater than 2, and in this case u will be a maximum or a minimum according as $\dfrac{d^2fx}{dx^2}$ is negative or positive: and if this last differential coefficient vanish, similar conclusions may be drawn from the next, and so on.

All this will appear clear from finding the value of u, corresponding to $x=a-h$, in a series by means of *Taylor's* Theorem.

CHAP. VIII.

On the Application of the Differential Calculus to Plane Curves referred to rectangular Co-ordinates.

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126. THE general equation of a plane curve referred to rectangular co-ordinates being

$$y = f(x), \text{ or } f(x, y) = 0,$$

it is manifest that the values of the successive differential coefficients of y, denoted by

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \&c., \frac{d^ny}{dx^n},$$

may be obtained in the same manner as those of u have been found in the preceding pages:

also if y_1, y_2, y_3, &c., y_m, be the values of y corresponding to $x + h$, $x + 2h$, $x + 3h$, &c., $x + mh$, we shall have by *Taylor's* Theorem,

$$y_1 = f(x + h) = y + \frac{dy}{dx}\frac{h}{1} + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \&c.;$$

$$y_2 = f(x + 2h) = y + \frac{dy}{dx}\frac{2h}{1} + \frac{d^2y}{dx^2}\frac{4h^2}{1.2} + \frac{d^3y}{dx^3}\frac{8h^3}{1.2.3} + \&c.;$$

$$y_3 = f(x + 3h) = y + \frac{dy}{dx}\frac{3h}{1} + \frac{d^2y}{dx^2}\frac{9h^2}{1.2} + \frac{d^3y}{dx^3}\frac{27h^3}{1.2.3} + \&c.;$$

&c...

$$y_m = f(x+mh) = y + \frac{dy}{dx}\frac{mh}{1} + \frac{d^2y}{dx^2}\frac{m^2h^2}{1.2} + \frac{d^3y}{dx^3}\frac{m^3h^3}{1.2.3} + \text{\&c.}:$$

and in the same manner, if the ordinates y_{-1}, y_{-2}, y_{-3}, &c. y_{-m} correspond to the abscissas $x-h$, $x-2h$, $x-3h$, &c., $x-mh$, we have

$$y_{-1} = f(x-h) = y - \frac{dy}{dx}\frac{h}{1} + \frac{d^2y}{dx^2}\frac{h^2}{1.2} - \frac{d^3y}{dx^3}\frac{h^3}{1.3.3} + \text{\&c.};$$

$$y_{-2} = f(x-2h) = y - \frac{dy}{dx}\frac{2h}{1} + \frac{d^2y}{dx^2}\frac{4h^2}{1.2} - \frac{d^3y}{dx^3}\frac{8h^3}{1.2.3} + \text{\&c.};$$

$$y_{-3} = f(x-3h) = y - \frac{dy}{dx}\frac{3h}{1} + \frac{d^2y}{dx^2}\frac{9h^2}{1.2} - \frac{d^3y}{dx^3}\frac{27h^3}{1.2.3} + \text{\&c.};$$

&c. ....

$$y_{-m} = f(x-mh) = y - \frac{dy}{dx}\frac{mh}{1} + \frac{d^2y}{dx^2}\frac{m^2h^2}{1.2} - \frac{d^3y}{dx^3}\frac{m^3h^3}{1.2.3} + \text{\&c.}:$$

also, from the equations above exhibited, we obtain

$$y_1 - y = \frac{dy}{dx}\frac{h}{1} + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \text{\&c.};$$

$$y_2 - y_1 = \frac{dy}{dx}\frac{h}{1} + \frac{d^2y}{dx^2}\frac{3h^2}{1.2} + \frac{d^3y}{dx^3}\frac{7h^3}{1.2.3} + \text{\&c.};$$

$$y_3 - y_2 = \frac{dy}{dx}\frac{h}{1} + \frac{d^2y}{dx^2}\frac{5h^2}{1.2} + \frac{d^3y}{dx^3}\frac{19h^3}{1.2.3} + \text{\&c.};$$

&c. ....

and

$$y_{-1} - y = -\frac{dy}{dx}\frac{h}{1} + \frac{d^2y}{dx^2}\frac{h^2}{1.2} - \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \text{\&c.};$$

$$y_{-2}-y_{-1}=-\frac{dy}{dx}\frac{h}{1}+\frac{d^2y}{dx^2}\frac{3h^2}{1.2}-\frac{d^3y}{dx^3}\frac{7h^3}{1.2.3}+\&c.;$$

$$y_{-3}-y_{-2}=-\frac{dy}{dx}\frac{h}{1}+\frac{d^2y}{dx^2}\frac{5h^2}{1.2}-\frac{d^3y}{dx^3}\frac{19h^3}{1.2.3}+\&c.;$$

&c. ....

which are the magnitudes of the differences of the successive ordinates corresponding to the given difference of abscissas h; and it is manifest that the difference of any other two values of y may be expressed in the same manner.

127. Cor. Since by the last article it appears that when h is supposed to be indefinitely small, the differences

$$y_1-y,\ y_2-y_1,\ y_3-y_2,\ \&c.,$$

are all equal to one another, and to $\frac{dy}{dx}h$, it follows from similar triangles, that an indefinitely small portion of a curve may be considered to possess the properties of a right line: whence if s represent the arc AP, we have

$$\frac{ds}{dx}=\text{limit of }\sqrt{\frac{PR^2+RQ^2}{PR^2}}$$

$$=\text{limit of }\sqrt{1+\left(\frac{dy}{dx}+\frac{d^2y}{dx^2}\frac{h}{1.2}+\&c.\right)^2}$$

$$=\sqrt{1+\frac{dy^2}{dx^2}},\text{ and }\therefore ds=dx\sqrt{1+\frac{dy^2}{dx^2}};$$

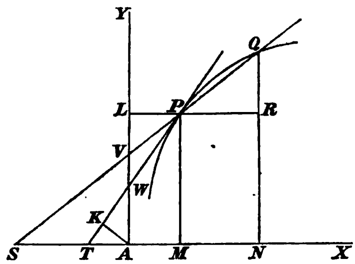
as has been already proved in a preceding article.

I. TANGENTS.

128. *To find the angles which a right line, cutting a curve in two points, makes with the co-ordinate axes.*

Let the right line $QPVS$ cut the curve APQ in the points P and Q, and the axes of x and y in S and V: draw the ordinates

MP, *NQ*, and let the straight line *LPR* be drawn parallel to



the axis of x, meeting the axis of y in L: also let

$$AM = x, \quad MP = y, \text{ and } MN = h;$$

then we shall have

$$\tan PSX = \tan QPR = \frac{QR}{PR}$$

$$= \frac{\frac{dy}{dx}\frac{h}{1} + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \&c.}{h}$$

$$= \frac{dy}{dx} + \frac{d^2y}{dx^2}\frac{h}{1.2} + \frac{d^3y}{dx^3}\frac{h^2}{1.2.3} + \&c. = \cot PVY;$$

which, by assigning given values to x, y and h, becomes known.

Ex. Let the curve be of the parabolic kind, whose equation is $ay = x^2$: then we have

$$\frac{dy}{dx} = \frac{2x}{a}, \quad \frac{d^2y}{dx^2} = \frac{2}{a}, \quad \frac{d^3y}{dx^3} = 0, \ \&c.:$$

whence is obtained

$$\tan PSX = \frac{2x}{a} + \frac{2}{a}\frac{h}{1.2} = \frac{2x + h}{a} = \cot PVY:$$

and if the points P and Q of the curve be given, or x and h be known, these angles become assignable.

129. *To find the angles which a right line, touching a curve at any point, makes with the co-ordinate axes.*

Let the right line PT touch the curve APQ at the point P, and cut the axes of x and y in the points T and W: then, in the last article, we have seen that

$$\tan PSX = \frac{dy}{dx} + \frac{d^2y}{dx^2}\frac{h}{1.2} + \frac{d^3y}{dx^3}\frac{h^2}{1.2.3} + \&c. = \cot PVY:$$

wherefore if h be diminished *sine limite*, it is manifest that the secant $QPVS$, by the coincidence of Q with P, then comes into the position of the tangent PT, and that the angle PSX then coincides with the angle PTX, and PVY with PWY;

$$\therefore \tan PTX = \frac{dy}{dx} = \cot PWY:$$

whence, if $y = f(x)$, be the equation to any proposed curve, the inclinations of a tangent, at a point whose co-ordinates are x and y, to the co-ordinate axes, may be found from the formula,

$$\tan PTX = \frac{dy}{dx} = \cot PWY.$$

Hence also, if at any point whose co-ordinates are x and y, we have $\frac{dy}{dx} = 0$, the tangent there will be parallel to the axis of x; and if $\frac{dy}{dx} = \infty$, it will be perpendicular to it.

Ex. 1. If the curve be a circle whose equation from the vertex is

$$y = \sqrt{2ax - x^2}, \text{ then will } \frac{dy}{dx} = \frac{a-x}{\sqrt{2ax-x^2}};$$

$$\text{whence } \tan PTX = \frac{a-x}{\sqrt{2ax-x^2}} = \cot PWY;$$

$$\text{or } \angle PTX = \tan^{-1}\frac{a-x}{\sqrt{2ax-x^2}}, \text{ and } PWY = \cot^{-1}\frac{a-x}{\sqrt{2ax-x^2}};$$

which therefore become known at any point where the value of x is assigned.

Ex. 2. Let $y^m-(ax+b)y^{m-1}+(cx^2+ex+f)y^{m-2}-\&c.=0$, be the general equation to a curve of m dimensions; and suppose all the values of y to be possible, and to be represented by y_1, y_2, y_3, &c., y_m; then, from the nature of equations, we have

$$y_1+y_2+y_3+\&c.+y_m=ax+b;$$

$$\text{whence } \frac{dy_1}{dx}+\frac{dy_2}{dx}+\frac{dy_3}{dx}+\&c.+\frac{dy_m}{dx}=a:$$

that is, the sum of the trigonometrical tangents of all the angles which tangents at the different points of a curve in which an ordinate meets it, make with the axis of x, is a constant magnitude.

130. Cor. 1. Since an arc and its tangent are coincident at the point of concourse, it follows that the inclinations of the curve to the axes at any point are the same as those of the tangent at that point; and therefore, if X and Y represent the angles of inclination of the curve to the axes of x and y respectively, we shall have

$$\tan X=\frac{dy}{dx}=\cot Y, \text{ and } \tan Y=\frac{dx}{dy}=\cot X.$$

Ex. Let $y=\frac{b}{a}\sqrt{2ax-x^2}$, which is the equation to an ellipse; then $\tan X=\frac{dy}{dx}=\frac{b}{a}\,\frac{a-x}{\sqrt{2ax-x^2}}=\cot Y$:

whence, if we assume $x=\frac{a}{2}$, or equal half the semi-axis major,

$$\text{we find } \tan X=\frac{b}{a\sqrt{3}}=\cot Y,$$

$$\text{or } X=\tan^{-1}\frac{b}{a\sqrt{3}}, \text{ and } Y=\cot^{-1}\frac{b}{a\sqrt{3}};$$

which give the inclinations of the elliptic arc, or of its tangent, to the co-ordinate axes at a point whose ordinate bisects the semi-axis major; also, if $b=a$, or the ellipse become a circle, we have

$$\tan X = \frac{1}{\sqrt{3}} = \tan 30^0, \text{ or } X = 30^0 \text{ and } \therefore\ Y = 60^0:$$

again, if $x=a$ the semi-axis major, we shall have

$$\tan X = 0, \text{ or } X = 0;$$

which proves that the tangent at the extremity of the axis minor makes no finite angle with the axis of x, and therefore, that the curve is there parallel to it.

131. Cor. 2. Conversely, if the angles of inclination be given, the corresponding points in the curve may be found.

For, if a denote the angle of inclination to the axis of x, we must obviously have

$$\tan a = \frac{dy}{dx} = f'(x):$$

and by the solution of this equation, the values of x, and therefore the positions of the required points will be found.

132. Cor. 3. Hence also, if the equations to any two curves, having the same or parallel axes, be

$$y = f(x) \text{ and } y' = F(x'),$$

and X and X' denote their inclinations to the axis of x at any points, the trigonometrical tangent of the inclination of their tangents at the points (x, y), (x', y') to each other is obviously

$$= \tan(X \sim X') = \frac{\tan X \sim \tan X'}{1 + \tan X \tan X'} = \frac{\dfrac{dy}{dx} \sim \dfrac{dy'}{dx'}}{1 + \dfrac{dy}{dx}\dfrac{dy'}{dx'}}.$$

133. *To find the angles in which a curve intersects the co-ordinate axes.*

Since the curve and its tangent at any point are coincident, it follows that the angles in which the curve cuts its axes, are the same as those formed by the tangents with the axes at the points of intersection:

hence, therefore, if $y=f(x)$, be the equation of the curve, and $f(x)$ be assumed $=0$, the roots of the equation so formed will determine the points in the axis of x where the curve meets it; and these quantities being substituted in the expression for $\frac{dy}{dx}$, will manifestly give the trigonometrical tangents of the required angles. Similarly for the axis of y.

Ex. 1. Let $y^2=mx+nx^2$, which is the equation of the conic sections; then assuming

$$y^2=mx+nx^2=0,$$

$$\text{we get } x=0 \text{ and } x=-\frac{m}{n}:$$

$$\text{also, } \frac{dy}{dx}=\frac{1}{2}\,\frac{m+2nx}{\sqrt{mx+nx^2}};$$

from which, if 0 be put for x, we find corresponding thereto

$$\tan X=\infty=\tan 90^\circ;$$

therefore $X=90^\circ$, which is the angle of intersection with the axis of x at the origin of the co-ordinates:

again, if $x=-\frac{m}{n}$, we shall obviously have

$$\tan X=-\infty=-\tan 90^\circ,\ \therefore X=-90^\circ:$$

and thence we conclude that the conic sections, when the relations between their co-ordinates is expressed by the equation

above given, cut the axis of x at right angles, and are therefore at the points of intersection parallel to the axis of y.

In the parabola, $n=0$, and therefore when

$$x=-\frac{m}{n}=-\infty\,, \text{ we have } \frac{dy}{dx}=0;$$

or the curve at an infinite distance becomes parallel to the axis of x.

Ex. 2. Let the proposed curve be expressed by the equation

$$y=a\sin x+b\cos x:$$

then, at the points where it meets the axis of x, we must have

$$a\sin x+b\cos x=0:$$

from which $x=\sin^{-1}\left(\pm\frac{b}{\sqrt{a^2+b^2}}\right)=\cos^{-1}\left(\mp\frac{a}{\sqrt{a^2+b^2}}\right)$:

also, corresponding to these values of x, we have

$$\begin{aligned}\frac{dy}{dx}&=a\cos x-b\sin x\\ &=\mp\frac{a^2}{\sqrt{a^2+b^2}}\mp\frac{b^2}{\sqrt{a^2+b^2}}\\ &=\mp\frac{a^2+b^2}{\sqrt{a^2+b^2}}=\mp\sqrt{a^2+b^2};\end{aligned}$$

or, the trigonometrical tangents of the angles in which this curve intersects the axis of x are $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$.

Also, to find the angles in which it meets the axis of y, we must have $x=0$, and therefore

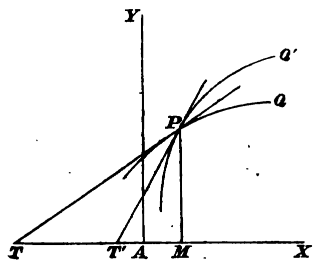
$$y=a\sin 0+b\cos 0=b:$$

and the tangent of the required angle

$$=\frac{1}{\frac{dy}{dx}}=\frac{1}{a\cos 0-b\sin 0}=\frac{1}{a}.$$

134. *To find the rectilineal angle in which two given curves intersect each other.*

Let $y=f(x)$ and $y'=F(x')$, be the equations to any two



curves PQ and PQ' referred to the same rectangular axes; then, at points of intersection, we must have $x'=x$ and $y'=y$; whence is obtained the equation

$$f(x)-F(x)=0,$$

the roots of which will manifestly determine the co-ordinates of the points of intersection:

also, if PT and PT' be tangents to the curves at P, we have

$$\tan TPT'=\tan(PT'X-PTX)$$

$$=\frac{\tan PT'X-\tan PTX}{1+\tan PT'X\tan PTX}$$

$$=\frac{\frac{dy'}{dx'}-\frac{dy}{dx}}{1+\frac{dy'}{dx'}\frac{dy}{dx}}:$$

wherein the differential coefficients $\frac{dy}{dx}$ and $\frac{dy'}{dx'}$, must be replaced by their values derived from the equations of the proposed curves, and corresponding to the values of x determined by the equation

$$f(x) - F(x) = 0.$$

Ex. 1. Let the parabola, whose equation is $y^2 = 4ax$, be intersected by the straight line determined by the equation $y' = x'$: then, if $x' = x$ and $y' = y$, we shall have

$$4ax = x^2 \text{ and } \therefore\ x = 0 \text{ and } x = 4a;$$

and thus the co-ordinates of the points of concourse are found:

$$\text{also, } \frac{dy}{dx} = \sqrt{\frac{a}{x}} \text{ and } \frac{dy'}{dx'} = 1;$$

$$\therefore \tan TPT' = \frac{\frac{dy'}{dx'} - \frac{dy}{dx}}{1 + \frac{dy'}{dx'}\frac{dy}{dx}} = \frac{\sqrt{x} - \sqrt{a}}{\sqrt{x} + \sqrt{a}};$$

which is -1 when $x = 0$, and $\frac{1}{3}$ when $x = 4a$:

that is, the straight line cuts the parabola at angles whose trigonometrical tangents are -1 and $\frac{1}{3}$.

Ex. 2. Let the curves be an ellipse and a circle whose equations are

$$y^2 = \frac{b^2}{a^2}(2ax - x^2) \text{ and } y'^2 = 2bx' - x'^2;$$

then, at the points where they meet each other, we have $x' = x$ and $y' = y$; whence

$$b\sqrt{2ax - x^2} = a\sqrt{2bx - x^2},$$

$$\text{which gives } x = 0 \text{ and } x = \frac{2ab}{a + b};$$

and thus we find the co-ordinates of the points of concourse:

$$\text{also, } \tan TPT' = \frac{\dfrac{dy'}{dx'} - \dfrac{dy}{dx}}{1 + \dfrac{dy'}{dx'}\dfrac{dy}{dx}}$$

$$= \frac{a(b-x)\sqrt{2ax-x^2} - b(a-x)\sqrt{2bx-x^2}}{a\sqrt{2ax-x^2}\sqrt{2bx-x^2} + b(a-x)(b-x)};$$

hence, when $x=0$, $\tan TPT'=0$, or $TPT'=0$; and therefore at the origin, the curves having a common tangent, form no angle with each other;

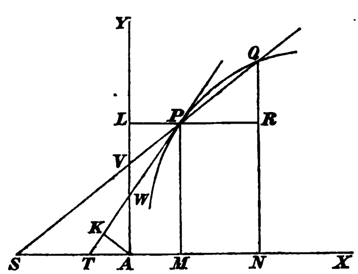
and when $x = \dfrac{2ab}{a+b}$, by substitution in the same expression, we find

$$\tan TPT' = -\frac{2(a-b)}{3a-b}\sqrt{\frac{a}{b}};$$

the negative sign shewing that the two tangents PT and PT' are situated with respect to each other on sides contrary to what they possess in the diagram.

135. *To determine the points in which the tangent to a curve intersects the co-ordinate axes.*

Let PWT be a tangent to the curve AP at the point P



whose co-ordinates are x and y, meeting the axes of x and y in T and W respectively:

$$\text{then } AT = TM - AM = MP \cot PTM - AM = \frac{y\,dx}{dy} - x:$$

$$\text{and } AW = AL - WL = MP - AM \cot PWL = y - \frac{x\,dy}{dx}:$$

and thus the distances of the points of intersection from the origin are found.

Also, the magnitudes of the lines MT and LW, which are called the *Subtangents* on the axes of x and y respectively, have been found above;

$$\text{for } MT = \frac{y\,dx}{dy} \text{ and } LW = \frac{x\,dy}{dx}.$$

136. Cor. 1. Hence, the position of the point P being given, and that of T or W being found from one of the expressions above investigated, the rectilineal tangent may be constructed by joining the points P and T, or P and W.

If $\frac{y\,dx}{dy} = \infty$, the corresponding subtangent is infinite, and the tangent becomes parallel to the axis of x; but if $\frac{y\,dx}{dy} = 0$, the tangent is parallel to the axis of y, and the ordinate becomes a tangent to the curve.

137. Cor. 2. The magnitude of the part PT of the tangent

$$= MP \operatorname{cosec} PTM = y\sqrt{1 + \cot^2 PTM} = y\sqrt{1 + \frac{dx^2}{dy^2}} = \frac{y\,ds}{dy};$$

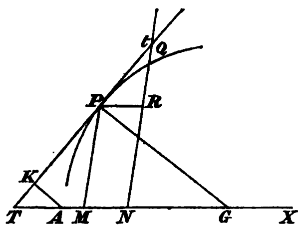
$$\text{similarly, we shall have } PW = x\sqrt{1 + \frac{dy^2}{dx^2}} = \frac{x\,ds}{dx}:$$

and if AK be drawn from the origin perpendicular to the tangent PT we have

$$PT : PM :: AT : AK, \text{ whence}$$

$$AK = \frac{y\,dx - x\,dy}{\sqrt{dx^2 + dy^2}} = \frac{y\,dx - x\,dy}{ds}.$$

138. If the ordinates be inclined to the axis of x at any



given angle, the expressions for the subtangents will remain precisely the same. For, we have, as above

$$\frac{MP}{MT} = \text{limit of } \frac{QR}{PR} = \text{limit of } \left\{\frac{\frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \&c.}{h}\right\}$$

$$= \text{limit of } \left(\frac{dy}{dx} + \frac{d^2y}{dx^2}\frac{h}{1.2} + \&c.\right) = \frac{dy}{dx};$$

$$\text{and } \therefore \text{ the subtangent } MT = \frac{y\,dx}{dy}, \text{ as before.}$$

Also, the values of PT and AK must then be found by the solutions of oblique-angled triangles.

Ex. 1. In an ellipse when the origin of co-ordinates is at the vertex, we have

$$y = \frac{b}{a}\sqrt{2ax - x^2}:$$

$$\therefore \log y = \log b - \log a + \tfrac{1}{2}\log(2ax - x^2):$$

$$\text{whence } \frac{dy}{dx} = \frac{(a-x)\,y}{2ax - x^2} = \frac{b}{a}\,\frac{(a-x)}{\sqrt{2ax - x^2}}:$$

$$\therefore AT = \frac{y\,dx}{dy} - x = \frac{2ax - x^2}{a - x} - x = \frac{ax}{a - x};$$

$$AW = y - \frac{xdy}{dx} = y - \frac{b}{a}\,\frac{(a-x)x}{\sqrt{2ax-x^2}} = b\sqrt{\frac{x}{2a-x}};$$

and thus the points in which the tangent cuts the co-ordinate axes are found.

Also, the subtangent on the axis of $x = \dfrac{ydx}{dy} = \dfrac{2ax-x^2}{a-x}$;

and the subtangent on the axis of $y = \dfrac{xdy}{dx} = \dfrac{b(ax-x^2)}{a\sqrt{2ax-x^2}}$.

Here it may be observed that the value of the subtangent on the axis of x, not involving b, will be the same whatever be the magnitude of the minor axis; and therefore if a circle, or any other ellipse be described upon the same axis major, the tangent at the point, in which the ordinate MP meets it, will cut the axis major produced in the point T.

The same may be proved of any two curves having common axes, and their ordinates in a given ratio.

Ex. 2. Find where the tangent to the parabolic curve whose equation is $y^m = ax^{m-1}$, meets the co-ordinate axes, and determine the magnitudes of the subtangents.

Here we have $m \log y = \log a + (m-1) \log x$:

$$\therefore \frac{mdy}{y} = \frac{(m-1)dx}{x} \text{ and } \frac{dy}{dx} = \frac{(m-1)y}{mx}:$$

$$\text{whence } AT = \frac{ydx}{dy} - x = \frac{mx}{m-1} - x = \frac{x}{m-1},$$

$$\text{and } AW = y - \frac{xdy}{dx} = y - \frac{m-1}{m}y = \frac{y}{m};$$

or, the tangent cuts off, from the co-ordinate axes, lines equal to $\dfrac{1}{m-1}$th and $\dfrac{1}{m}$th parts of the abscissa and ordinate respectively:

$$\text{also, the subtangent } MT = \frac{ydx}{dy} = \frac{m}{m-1}x,$$

$$\text{and the subtangent } LW = \frac{xdy}{dx} = \frac{m-1}{m}y:$$

and by means of any of these lines thus found, the rectilineal tangent to the curve may be constructed.

If $m = 2$, we have

$$MT = 2x \text{ and } LW = \tfrac{1}{2}y,$$

for the *Conical* parabola.

If $m = 3$, we find

$$MT = \tfrac{3}{2}x \text{ and } LW = \tfrac{2}{3}y,$$

for the *Cubical* parabola.

If $m = \frac{3}{2}$, we obtain

$$MT = 3x \text{ and } LW = \tfrac{1}{3}y,$$

for the *Semi-cubical* parabola: and so on.

Ex. 3. From the equation to hyperbolic curves

$$xy^m = a^{m+1},$$

we have $m \log y = (m+1) \log a - \log x$:

$$\therefore \frac{mdy}{y} = -\frac{dx}{x} \text{ and } \frac{dy}{dx} = -\frac{my}{x};$$

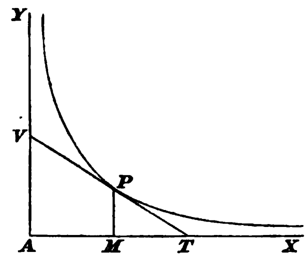
$$\text{whence } AT = \frac{ydx}{dy} - x = -\frac{x}{m} - x = -\frac{m+1}{m}x,$$

$$\text{and } AW = y - \frac{xdy}{dx} = y + my = (m+1)y:$$

$$\text{also } MT = -mx \text{ and } LW = -my,$$

which are the respective subtangents on the axes of x and y.

Whence if $m=1$, we have in the conical hyperbola between the asymptotes



$$AT=2x=2AM \text{ and } AV=2y=2MP:$$

and therefore area of the

$$\Delta\, ATV=\tfrac{1}{2}AT\,.\,AV=2xy=2a^2,$$

a constant magnitude entirely independent of the position of the point P.

Ex. 4. In the cissoid of *Diocles*, $y=\dfrac{x^{\frac{3}{2}}}{\sqrt{2a-x}}$; whence

$$\log y=\frac{3}{2}\log x-\frac{1}{2}\log(2a-x), \text{ and } \frac{dy}{ydx}=\frac{3a-x}{x(2a-x)};$$

$$\text{therefore the subtangent } MT=\frac{x(2a-x)}{3a-x};$$

and thus, to any point whose co-ordinates are assigned, the rectilineal tangent may be drawn.

If $x=0$, we have $y=0$, and therefore $\dfrac{dy}{dx}=0$ and the subtangent $=0$: therefore the axis of x is a tangent at the origin: if $x=2a$, we have $y=\infty$, $\therefore \dfrac{dy}{dx}=\infty$ and the subtangent $=0$: that is, the infinite ordinate is a tangent to the curve at an infinite distance.

Conversely, if the magnitude of the subtangent be given and $= \frac{a}{2}$, for instance, we shall have

$$\frac{x(2a-x)}{3a-x} = \frac{a}{2}; \text{ or } 2x^2 - 5ax + 3a^2 = 0,$$

from which, we find $x = a$ and $x = \frac{3}{2}a$; whence the two points to which this value of the subtangent belongs are determined, the ordinate of the former bisecting the diameter of the generating circle, and the latter the further radius.

Ex. 5. In the common cycloid whose equation is

$$y = \sqrt{2ax - x^2} + a \text{ vers}^{-1} \frac{x}{a},$$

$$\text{we have } \frac{dy}{dx} = \frac{(2a-x)}{\sqrt{2ax-x^2}} = \sqrt{\frac{2a-x}{x}};$$

$$\therefore \text{ the subtangent } MT = \frac{ydx}{dy} = \frac{xy}{\sqrt{2ax-x^2}};$$

whence, if AQ the corresponding chord of the generating circle be drawn, we have

$$MT = \frac{AM \,.\, MP}{MQ} \text{ or } \frac{MT}{MP} = \frac{AM}{MQ};$$

$\therefore$ PT is parallel to AQ, or the tangent of the cycloid is parallel to the corresponding chord of the generating circle.

Ex. 6. Let $y^3 - 3axy + x^3 = 0$, be the equation of the curve proposed; then, we have

$$y^2 dy - axdy - aydx + x^2 dx = 0,$$

$$\text{from which we get } \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax};$$

whence the subtangent $MT = \frac{ydx}{dy} = \frac{y^3 - axy}{ay - x^2} = \frac{2axy - x^3}{ay - x^2}$,

which involves both x and y; and therefore, if any value be assigned to x, the corresponding value of y must be determined before the tangent can be drawn.

Ex. 7. Let it be required to draw a tangent to the catenary whose differential equation is $sdy = adx$.

$$\text{Here, } AT = \frac{ydx}{dy} - x = \frac{ys}{a} - x = \frac{ys - ax}{a},$$

$$\text{and } AW = y - \frac{xdy}{dx} = y - \frac{ax}{s} = \frac{ys - ax}{s}:$$

also, we have seen that for the subtangents,

$$MT = \frac{ys}{a} \text{ and } LW = \frac{ax}{s};$$

and by means of any one of these expressions, a tangent may be drawn to any assigned point of the proposed curve.

Ex. 8. Let $y^m - (ax + b)y^{m-1} + \text{\&c.} + (kx^m + lx^{m-1} + \text{\&c.}) = 0$; then if y_1, y_2, y_3, &c., y_m, be the values of the different ordinates corresponding to the same abscissa x, we have, by the nature of equations,

$$y_1 \cdot y_2 \cdot y_3 \cdot \text{\&c. } y_m = kx^m + lx^{m-1} + \text{\&c.}:$$

$$\therefore \log y_1 + \log y_2 + \log y_3 + \text{\&c.} + \log y_m$$
$$= \log(kx^m + lx^{m-1} + \text{\&c.});$$

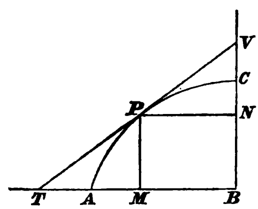
whence we obtain immediately,

$$\frac{dy_1}{y_1 dx} + \frac{dy_2}{y_2 dx} + \frac{dy_3}{y_3 dx} + \text{\&c.} + \frac{dy_m}{y_m dx}$$
$$= \frac{mkx^{m-1} + (m-1)lx^{m-2} + \text{\&c.}}{kx^m + lx^{m-1} + \text{\&c.}},$$

which is therefore the sum of the reciprocals of the subtangents on the axis of x:

also, if x be given, the last term $kx^m + lx^{m-1} + \&c.$ of the proposed equation remains the same, at whatever angle the ordinate is inclined to the line of abscissæ, and therefore the sum of the reciprocals of the subtangents at the points of intersection by any ordinate corresponding to a given abscissa is a constant quantity independent of the inclination of the co-ordinate axes, as appears from the last article.

139. If in any mixtilinear figure ABC contained by two given co-ordinates AB, BC, and the curve AC concave or convex to the axis, there be inscribed a rectangle $MPNB$,



and $AB = a$, $AM = x$, $MP = y$, then will its area $= y\,(a - x)$: now, this will be a *maximum* when u being equal to $y\,(a - x)$,

$$\text{we have } \frac{du}{dx} = (a - x)\,\frac{dy}{dx} - y = 0\,;$$

$$\therefore\ \frac{ydx}{dy} = a - x, \text{ or } MT = MB\,; \text{ and similarly } NV = NB:$$

that is, the sides of the greatest rectangle are equal to the corresponding subtangents belonging to the point P.

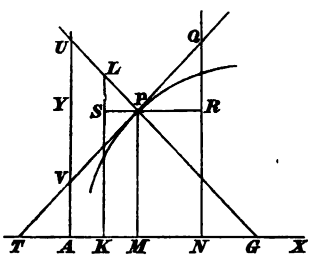
Hence also, the area of the triangle cut off by the tangent TPV to the same point P which therefore bisects it, is a *minimum* and equal to twice that of the rectangle.

After a similar manner it may be proved that the content of the cylinder generated by the revolution of an inscribed rectangle about the axis of x will be the *greatest possible*, when the altitude of the cylinder is equal to half the corres-

ponding subtangent; that the content of the cone generated by the corresponding triangle will be the least possible; and that the cylinder will be to the cone as 4 : 9.

140. *To find the equation to the tangent at any point of a proposed curve.*

Let AP be the curve, whose co-ordinates at the point P are $AM = x$, $MP = y$; QPT a tangent at P whose co-ordinates



at the point Q are $AN = x'$, $NQ = y'$; then, we shall have

$$\frac{QR}{PR} = \tan QPR = \tan PTM = \frac{dy}{dx}, \text{ or } QR = \frac{dy}{dx} PR;$$

wherefore $y' - y = \frac{dy}{dx}(x' - x)$, is the equation required.

Hence, if known values be assigned to x and y, or the position of the point P be given, the relation between x' and y' will be expressed by a simple equation; and if the line to which this equation belongs be drawn, it will be a tangent to the curve at the point proposed.

141. Cor. 1. Since the equation to the tangent line QPT is

$$y' - y = \frac{dy}{dx}(x' - x),$$

if we suppose $x = 0$, or the tangent cut the axis of x, we shall have

$$y' - y = -\frac{x\,dy}{dx}, \text{ and } \therefore\ y' \text{ or } AV = y - \frac{x\,dy}{dx};$$

but if we make $y'=0$, or the tangent intersect the axis of x,

$$\text{then will } -y = \frac{dy}{dx}(x'-x), \text{ and } \therefore\ -x' \text{ or } AT = \frac{y\,dx}{dy} - x:$$

hence also, since $y = \frac{dy}{dx}(x-x')$, we have in this case,

$\frac{y\,dx}{dy} = x - x' = AM + AT = MT$, the subtangent, as before.

142. Cor. 2. If we assume $x=0$ and $y=0$, and $-\frac{dx}{dy}$ be substituted in the place of $\frac{dy}{dx}$ in the equation to the tangent, we shall readily find the equation to the perpendicular upon the tangent from the origin of the co-ordinates to be $y' = -\frac{dx}{dy}x'$, or $y'\frac{dy}{dx} + x' = 0$.

Ex. 1. Let $y^2 = 4ax$; then $y = 2a^{\frac{1}{2}}x^{\frac{1}{2}}$ and $\frac{dy}{dx} = \sqrt{\frac{a}{x}}$; $\therefore$ the equation to the tangent of a common parabola is

$$y' - y = \sqrt{\frac{a}{x}}(x'-x),$$

$$\text{and } \therefore\ y' = 2a^{\frac{1}{2}}x^{\frac{1}{2}} + \sqrt{\frac{a}{x}}(x'-x)$$

$$= \frac{2a^{\frac{1}{2}}x + a^{\frac{1}{2}}x' - a^{\frac{1}{2}}x}{\sqrt{x}} = (x+x')\sqrt{\frac{a}{x}}:$$

and to construct the line expressed by this equation, let $x'=0$, whence we obtain $y' = \sqrt{ax} = \frac{1}{2}y$:
that is, take $AV = \frac{1}{2}MP$, and V is a point in the tangent, join PV, and this line produced will be the tangent required.

If $x = a$, we find the equation to the tangent of a common parabola at the extremity of the *latus rectum* to be $y' = a + x'$.

Since $PM = 2AV$ by construction, we have $MT = 2AT = 2AM$, as already proved in the second example of article (138).

Ex. 2. Let $y = \frac{a^2}{x}$; then $\frac{dy}{dx} = -\frac{a^2}{x^2}$, and the equation to the tangent of a common hyperbola between the asymptotes is

$$y' - y = -\frac{a^2}{x^2}(x' - x),$$

$$\therefore\ y' = \frac{a^2}{x} - \frac{a^2 x'}{x^2} + \frac{a^2}{x} = (2x - x')\frac{a^2}{x^2} \text{ and } yx' + xy' = 2a^2:$$

whence to construct it, let $x' = 0$, $\therefore\ y, = 2y$; that is, if AV be taken $= 2MP$, the line VP produced is the tangent to the curve.

143. *To find the equation to the tangent of a proposed curve, which shall make given angles with the axes.*

Let the required equation be $y' - y = \frac{dy}{dx}(x' - x)$, and the tangent of the given angle made with the axis of x be c; then we must have

$$\frac{dy}{dx} = f'(x) = c,$$

from which the values of x, and therefore of y may be found; whence, the relation between the co-ordinates of the required tangent becomes known, and it may therefore be constructed.

This is manifestly the same as to draw a tangent to a curve, which shall be parallel to a given straight line.

Ex. Let the equation $y = \frac{x}{a}\sqrt{a^2 - x^2}$, represent a curve, to which it is required to find the equation of a tangent making an angle of 45° with the axis of x; then we have

$$\frac{dy}{dx} = \frac{a^2 - 2x^2}{a\sqrt{a^2 - x^2}} = \tan 45° = 1,$$

whence are obtained $x=0$ and $x=\pm\frac{1}{2}a\sqrt{3}$:

in the former case $y=0$, and therefore the equation to the tangent is $y'=x'$: and in the latter we shall find $y=\pm\frac{1}{4}a\sqrt{3}$, and the equation becomes

$$y'=x'\mp\frac{1}{4}a\sqrt{3}:$$

and hence these tangents may be constructed as before.

144. *To find the equation to the tangent of a proposed curve, which shall pass through a given point.*

Let $y'-y=\frac{dy}{dx}(x'-x)$, be the required equation, α and β the co-ordinates of the given point; then since the tangent passes through this point, we must have

$$\beta-y=\frac{dy}{dx}(\alpha-x);$$

which, since y and $\frac{dy}{dx}$ are both functions of x, involves only one unknown quantity x: this equation being solved gives the values of x corresponding to the points of contact, and therefore of y and $\frac{dy}{dx}$; and thence the relation between x' and y' in the equation

$$y'-y=\frac{dy}{dx}(x'-x),$$

becomes known.

Ex. Let the curve be a parabola whose equation is $y^2=4ax$;

then since $\frac{dy}{dx}=\sqrt{\frac{a}{x}}$, we have $\beta=(x+\alpha)\sqrt{\frac{a}{x}}$;

whence $\beta^2x=a(x+\alpha)^2=ax^2+2a\alpha x+a\alpha^2$:

and by the solution of this quadratic equation, we find

$$x = \frac{\beta^2 - 2aa \pm \sqrt{\beta^4 - 4aa\beta^2}}{2a}:$$

therefore $y = 2\sqrt{ax}$ and $\frac{dy}{dx} = \sqrt{\frac{a}{x}}$, both become known by substitution, and consequently the equation to the tangent

$$y' - y = \frac{dy}{dx}(x' - x),$$

exhibits the relation between y' and x' and constant quantities: and since x has two unequal values, two rectilineal tangents to the parabola may be drawn through the same given point.

If $\beta^2 = 4aa$, the two values of x become equal to one another and to a, and the equation is then

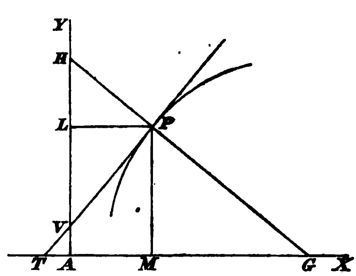
$$y' = (x' + a)\sqrt{\frac{x}{a}},$$

which is that of a tangent to the parabola at a point whose abscissa is a, as before shown in the first example of (142).

II. NORMALS.

145. *To find the points in which the normal to a proposed curve intersects the co-ordinate axes.*

Let PT be a tangent to the curve AP at the point P;



draw HPG perpendicular to it meeting the axes of x and y in G and H respectively: then is HPG called a *normal* line, and we have

$$AG = AM + MG = AM + MP \tan MPG$$
$$= x + y \tan PTM$$
$$= x + \frac{y\,dy}{dx}:$$

and
$$AH = AL + LH = AL + PL \tan LPH$$
$$= y + x \cot PTM$$
$$= y + \frac{x\,dx}{dy}:$$

wherefore the points G and H are ascertained corresponding to any assigned values of x and y.

Also, the *Subnormal* on the axis of x, or $MG = \frac{y\,dy}{dx}$:

and the *Subnormal* on the axis of y, or $LH = \frac{x\,dx}{dy}$:

and therefore if the point P be given, or the values of x and y be assigned, and MG or LH be made equal to its value thus deduced, the line PG or PH which will be a normal to the curve at P may be constructed.

Conversely, if either of the subnormals be given, the point in the curve may be determined.

146. Cor. 1. Hence, the lengths of the lines PG and PH which are called the normals belonging to the axes of x and y respectively, may be ascertained.

For, $PG = PM \sec MPG$

$$= PM\sqrt{1 + \tan^2 PTM} = y\sqrt{1 + \frac{dy^2}{dx^2}} = \frac{y\,ds}{dx}:$$

and $PH = PL \sec LPH$

$$= PL\sqrt{1 + \cot^2 PTM} = x\sqrt{1 + \frac{dx^2}{dy^2}} = \frac{x\,ds}{dy}.$$

Also, the angles of inclination of the normal line HPG to the co-ordinate axes may easily be found.

$$\text{For, } \tan PGX = -\tan PGM = -\cot PTM = -\frac{dx}{dy};$$

$$\text{and } \tan PHY = -\tan PHL = -\tan PTM = -\frac{dy}{dx}:$$

from which the angles themselves may be determined.

If the ordinates be inclined to the axes, expressions for determining the values of the subnormal MG, the normal PG and the tangent of the angle PGX in the diagram of (138), must be found by the solutions of oblique-angled triangles.

147. Cor. 2. The normal is in general the shortest or longest straight line that can be drawn from a given point to a curve.

For, let a, β be the co-ordinates of the given point, x, y those of any point in the curve: then, if u be the distance between these points, it is manifest that we shall have

$$u = \sqrt{(a-x)^2 + (\beta - y)^2}:$$

$$\therefore \frac{du}{dx} = -\frac{(a-x) + (\beta - y)\dfrac{dy}{dx}}{\sqrt{(a-x)^2 + (\beta - y)^2}} = 0,$$

by the nature of maxima and minima, as appears from (110): whence we have $(a - x) + \beta\dfrac{dy}{dx} = \dfrac{ydy}{dx}$; and therefore the least or greatest straight line is a normal to the curve, because the subnormal $\dfrac{ydy}{dx}$ is of the proper magnitude, as will readily appear upon examination of the circumstances of the point L in the diagram of (140).

Ex. 1. The general equation to the conic sections referred to their principal axes being

$$y^2 = mx + nx^2,$$

we shall have for them,

$$AG = x + \frac{y\,dy}{dx} = x + \tfrac{1}{2}m + nx = \tfrac{1}{2}m + (n+1)\,x;$$

$$AH = y + \frac{x\,dx}{dy} = y + \frac{2xy}{m+2nx} = \frac{m+2(n+1)x}{m+2nx}\,y;$$

whence the positions of the points G and H are determined:

also, $MG = \dfrac{y\,dy}{dx} = \tfrac{1}{2}m + nx$, and $LH = \dfrac{x\,dx}{dy} = \dfrac{2xy}{m+2nx}$;

which are expressions for the subnormals on the axes of x and y respectively.

If $n=0$, we shall have in the common parabola,

$$AG = x + \tfrac{1}{2}m \text{ and } AH = \frac{m+2x}{m}\,y:$$

also, the subnormal on the axis of x, or $MG = \tfrac{1}{2}m$ a constant magnitude, and that on the axis of y, or

$$LH = \frac{2xy}{m} = \frac{2x^{\frac{3}{2}}}{\sqrt{m}}.$$

If n be negative, the corresponding conic section is an ellipse;

$$\therefore\ AG = \tfrac{1}{2}m - (n-1)x,\ AH = \frac{m-2(n-1)x}{m-2nx}\,y:$$

$$MG = \tfrac{1}{2}m - nx \quad \text{and } LH = \frac{2xy}{m-2nx}.$$

If n be positive, the corresponding curve is an hyperbola.

Ex. 2. Let $y^m - (ax+b)\,y^{m-1} + (cx^2+ex+f)\,y^{m-2} - \&c. = 0$; then, if y_1, y_2, y_3, &c. y_m, be the different values of the or-

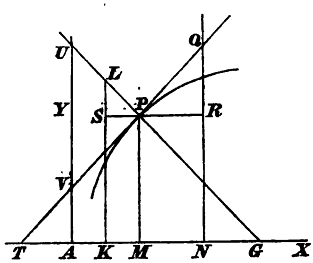
dinates belonging to the same abscissa x, we shall have, by the nature of equations,

$$y_1^2+y_2^2+y_3^2+\&c.+y_m^2=(ax+b)^2-2(cx^2+ex+f)$$
$$=(a^2-2c)x^2+2(ab-e)x+b^2-2f:$$
$$\therefore \frac{y_1dy_1}{dx}+\frac{y_2dy_2}{dx}+\frac{y_3dy_3}{dx}+\&c.+\frac{y_mdy_m}{dx}$$
$$=(a^2-2c)x+ab-e;$$

which is the sum of the subnormals on the axis of x of a curve of m dimensions corresponding to the points of intersection made by an ordinate whose abscissa is x; and if $a^2=2c$, this sum will be of the same magnitude whatever be the value of x.

148. *To find the equation to the normal at any point of a proposed curve.*

Let PG be a normal to the curve AP at P; $AM=x$, $MP=y$, the co-ordinates of the point P of the curve, and



$AK=x_,$, $KL=y_,$ those of the point L of the normal; draw PS parallel to AG: then, we have

$$\frac{PS}{SL}=\frac{\cos SPL}{\sin SPL}=\tan PTM=\frac{dy}{dx}:$$

whence we obtain $SL=\dfrac{dx}{dy}SP$, or $y_,-y=-\dfrac{dx}{dy}(x_,-x)$,

which is the equation to the normal line UPG.

By assigning to x and y given values, the relation between $x_{,}$ and $y_{,}$ becomes known, and the normal may be constructed by drawing the line to which this equation belongs.

Ex. 1. Let $y=\sqrt{2ax-x^2}$; then $\dfrac{dy}{dx}=\dfrac{a-x}{\sqrt{2ax-x^2}}$;

$$\text{whence } y_{,}-y=-\frac{\sqrt{2ax-x^2}}{a-x}(x_{,}-x)$$

$$=\frac{x\sqrt{2ax-x^2}}{a-x}-\frac{x_{,}\sqrt{2ax-x^2}}{a-x};$$

$$\text{and } \therefore\ y_{,}=\sqrt{2ax-x^2}+\frac{x\sqrt{2ax-x^2}}{a-x}-\frac{x_{,}\sqrt{2ax-x^2}}{a-x}$$

$$=\frac{a-x_{,}}{a-x}\sqrt{2ax-x^2};$$

which is the equation to the normal of a circle: and if $y_{,}=0$, we have $a-x_{,}=0$, or $x_{,}=a$, whatever be the value of x; and therefore the radius of the circle is always a normal to it.

Ex. 2. Let $y=\dfrac{x^{\frac{3}{2}}}{\sqrt{2a-x}}$; then $\dfrac{dy}{dx}=\dfrac{x^{\frac{1}{2}}(3a-x)}{(2a-x)^{\frac{3}{2}}}$;

$$\therefore\ y_{,}-y=-\frac{(2a-x)^{\frac{3}{2}}}{x^{\frac{1}{2}}(3a-x)}(x_{,}-x)=\frac{x(2a-x)^{\frac{3}{2}}-x_{,}(2a-x)^{\frac{3}{2}}}{x^{\frac{1}{2}}(3a-x)};$$

$$\text{and } \therefore\ y_{,}=\frac{x^2(3a-x)+x(2a-x)^2-x_{,}(2a-x)^2}{(3a-x)\sqrt{2ax-x^2}}$$

$$=\frac{ax(4a-x)-x_{,}(2a-x)^2}{(3a-x)\sqrt{2ax-x^2}};$$

the equation to the normal of the cissoid of *Diocles*, which is therefore easily constructed.

If $x=a$, and therefore $y=a$, the equation to the normal of this curve at the point where it cuts the generating circle, will be

$$y_{,}=\frac{1}{2}(3a-x_{,});$$

also if $y_{,}$ be made $=0$, we find $x_{,}=3a$; and consequently the line intercepted between the vertex of the curve and this normal is trisected by the centre and by the further extremity of the diameter, of the generating circle.

149. Cor. In the equation to the normal just found,

$$y_{,}-y=-\frac{dx}{dy}(x_{,}-x),$$

if we make $x=0$ and $y=0$, and substitute $-\frac{dy}{dx}$ in the place of $\frac{dx}{dy}$, we shall find the equation to the straight line drawn from the origin perpendicular to the normal to be

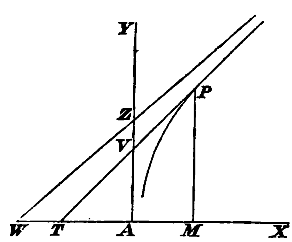
$$y_{,}=\frac{x_{,}dy}{dx}\text{, or } y_{,}-\frac{dy}{dx}x_{,}=0.$$

150. The methods pursued in articles (143) and (144) will enable us to find the equations to normals making given angles with the axes or passing through given points, and consequently to draw them.

III. ASYMPTOTES.

151. *To find whether a proposed curve admits of a rectilineal asymptote, and to determine its position.*

Let AP be any curve whereof the co-ordinates to the point P are $AM=x$, $MP=y$; and let PT be a tangent at the



point P meeting the axes of x and y in T and V; then we have seen in (135) that

$$AT = \frac{y\,dx}{dy} - x \text{ and } AV = y - \frac{x\,dy}{dx}:$$

now if both of the lines AT and AV do not become infinite when one or both of the co-ordinates x and y are indefinitely increased, the tangent PT is said to become a *rectilineal asymptote* to the curve:

also, if AW and AZ be the ultimate values of the lines AT and AV, when one or both of the co-ordinates are increased *sine limite*, two points W and Z are found, through which the asymptote must pass, and consequently WZ produced will be the line required.

(α) If AW be finite, and AZ infinite, the asymptote will be perpendicular to the axis of x.

(β) If AW be infinite, and AZ finite, the asymptote is parallel to the axis of x.

(γ) If AW be infinite and AZ vanish, the axis of x is an asymptote to the curve.

(δ) If AW and AZ both vanish, the asymptote passes through the origin of the co-ordinates; and since only one point in the line is thus determined, its direction must be found by seeking the value of $\tan X = \frac{dy}{dx}$, when one or both of the co-ordinates become infinite.

Ex. 1. In the conic sections whose *latus rectum* is m,

we have $y^2 = mx + nx^2$; then will

$$AT = \frac{y\,dx}{dy} - x = \frac{2mx + 2nx^2}{m + 2nx} - x = \frac{mx}{m + 2nx} = \frac{m}{\frac{m}{x} + 2n};$$

$$\text{and } AV = y - \frac{x\,dy}{dx} = \frac{mx}{2\sqrt{mx + nx^2}} = \frac{m}{2\sqrt{\frac{m}{x} + n}}:$$

hence, if x be indefinitely great, we have $\frac{m}{x}=0$, and therefore

$$AW=\frac{m}{2n} \text{ and } AZ=\frac{m}{2\sqrt{n}};$$

that is, if AW and AZ be taken equal to these values respectively, the curve being an hyperbola admits of an asymptote whose position is determined by joining the points W and Z.

If $n=0$, the curve is a parabola, $AW=\infty$ and $AZ=\infty$, and there can be no asymptote at a finite distance.

If n be negative, or the curve be an ellipse, the value of AZ is impossible, and therefore it does not admit of an asymptote.

Ex. 2. Let $y=\frac{x^{\frac{3}{2}}}{\sqrt{2a-x}}$; then $\frac{ydx}{dy}=\frac{x(2a-x)}{3a-x}$;

whence $AT=\frac{ydx}{dy}-x=\frac{x(2a-x)}{3a-x}-x=-\frac{ax}{3a-x}$;

$$AV=y-\frac{xdy}{dx}=\frac{x^{\frac{3}{2}}}{\sqrt{2a-x}}-\frac{xy(3a-x)}{x(2a-x)}=-\frac{(a-x)x^{\frac{3}{2}}}{(2a-x)^{\frac{3}{2}}};$$

now, since the value of y becomes impossible when x is greater than $2a$, if this curve admit of an asymptote, we must have $y=\infty$ and therefore $x=2a$; whence

$$AW=-\frac{2a^2}{3a-2a}=-2a \text{ and } AZ=\frac{(2a-a)(2a)^{\frac{3}{2}}}{(2a-2a)^{\frac{3}{2}}}=\infty;$$

and therefore the extreme ordinate is an asymptote to the cissoid, as appears from (a).

Ex. 3. Let $y=\frac{x}{1+x^2}$; wherefore $\frac{ydx}{dy}=\frac{x(1+x^2)}{1-x^2}$:

$$\therefore AT=\frac{ydx}{dy}-x=\frac{x(1+x^2)}{1-x^2}-x=\frac{2x^3}{1-x^2}=\frac{2}{\frac{1}{x^3}-\frac{1}{x}};$$

and $AV = y - \frac{xdy}{dx} = \frac{x}{1+x^2} - \frac{x(1-x^2)}{(1+x^2)^2} = \frac{2x^3}{(1+x^2)^2}$:

let now x be indefinitely increased, in order to find the ultimate values of AT and AV; then $AW = \infty$ and $AZ = 0$ from which we conclude by (γ) that the axis of x indefinitely extended is an asymptote to the curve.

Ex. 4. Let $\sqrt{y} = \frac{a}{\sqrt{x}} + \sqrt{x}$; then, $y = \frac{a^2}{x} + 2a + x$,

$$\text{and } \frac{dy}{dx} = -\frac{a^2}{x^2} + 1 = \frac{x^2 - a^2}{x^2}; \text{ wherefore}$$

$$AT = \frac{ydx}{dy} - x = \frac{(x+a)x}{x-a} - x = \frac{2ax}{x-a} = \frac{2a}{1 - \frac{a}{x}};$$

$$AV = y - \frac{xdy}{dx} = \frac{a^2 + 2ax + x^2}{x} - \frac{x^2 - a^2}{x} = \frac{2a^2 + 2ax}{x}:$$

first, let $x = \infty$, then $y = \infty$, $AW = 2a$ and $AZ = 2a$:

next, let $y = \infty$, then $x = 0$, $AW = 0$ and $AZ = \infty$:

hence this curve admits of two asymptotes, one making an angle of 45^0 with the axis of x, and the other perpendicular to it and passing through the origin of the co-ordinates.

Ex. 5. Let $y^3 - 3axy + x^3 = 0$; then, $\frac{dx}{dy} = \frac{ax - y^2}{x^2 - ay}$:

whence

$$AT = \frac{ydx}{dy} - x = \frac{axy - y^3}{x^2 - ay} - x = \frac{2axy - x^3 - y^3}{x^2 - ay} = -\frac{axy}{x^2 - ay};$$

$$AV = y - \frac{xdy}{dx} = y - \frac{x^3 - axy}{ax - y^2} = \frac{2axy - x^3 - y^3}{ax - y^2} = -\frac{axy}{ax - y^2};$$

and to obtain the values of these expressions, when x is made indefinitely great, assume $y=tx$, and therefore

$$t^3x^3-3atx^2+x^3=0, \text{ or } x=\frac{3at}{1+t^3};$$

hence, if $x=\infty$, $1+t^3=0$, or $t=-1$ and therefore $y=-x$: whence we find

$$AW=-\frac{axy}{x^2-ay}=\frac{ax^2}{x^2+ax}=\frac{a}{1+\frac{a}{x}}=a;$$

$$\text{and } AZ=-\frac{axy}{ax-y^2}=\frac{ax^2}{ax-x^2}=\frac{a}{\frac{a}{x}-1}=-a,$$

and consequently the required asymptote cuts the axes of x and y at angles equal to each other.

Ex. 6. Let $ax^4+4c^3xy-by^4=0$; then $\frac{dx}{dy}=\frac{by^3-c^3x}{ax^3+c^3y}$:

$$\text{whence } AT=\frac{ydx}{dy}-x=\frac{by^4-c^3xy}{ax^3+c^3y}-x=\frac{2c^3xy}{ax^3+c^3y};$$

$$\text{and } AV=y-\frac{xdy}{dx}=y-\frac{ax^4+c^3xy}{by^3-c^3x}=\frac{2c^3xy}{by^3-c^3x};$$

and to determine the limits of these quantities, let $y=tx$, and therefore

$$ax^4+4c^3tx^2-bt^4x^4=0, \text{ or } (a-bt^4)x^2+4c^3t=0,$$

wherefore $x=\sqrt{\frac{4c^3t}{bt^4-a}}$, which will evidently become infinite when $t=\frac{a^{\frac{1}{4}}}{b^{\frac{1}{4}}}$; whence is obtained $y=\frac{a^{\frac{1}{4}}}{b^{\frac{1}{4}}}x$, and we have

$$AW = \frac{2\dfrac{a^{\frac{1}{4}}}{b^{\frac{1}{4}}}c^3x^2}{ax^3 + \dfrac{a^{\frac{1}{4}}}{b^{\frac{1}{4}}}c^3x} = 0; \quad AZ = \frac{2\dfrac{a^{\frac{1}{4}}}{b^{\frac{1}{4}}}c^3x^2}{\dfrac{a^{\frac{3}{4}}}{b^{\frac{3}{4}}}bx^3 - c^3x} = 0:$$

that is, the asymptote passes through the origin of the co-ordinates: and to determine its position, we have

$$\frac{dy}{dx} = \frac{ax^3 + c^3y}{by^3 - c^3x} = \frac{ax^3 + \dfrac{a^{\frac{1}{4}}}{b^{\frac{1}{4}}}c^3x}{\dfrac{a^{\frac{3}{4}}}{b^{\frac{3}{4}}}bx^3 - c^3x} = \frac{a^{\frac{1}{4}}}{b^{\frac{1}{4}}} = \tan X:$$

and hence the asymptote may be drawn.

152. *To find the equation to the rectilineal asymptote of a curve.*

Since the equation to the tangent has been found in article (140) to be

$$y' - y = \frac{dy}{dx}(x' - x),$$

if one or both of the co-ordinates x and y of the curve be made infinite, the equation of the asymptote will be determined: and the equation of the asymptote being found, the line itself may be drawn.

Ex. Let $y^m = ax^{m-1} + x^m$, be the equation of the proposed curve; then

$$\frac{dy}{dx} = \frac{(m-1)\,ax^{m-2} + mx^{m-1}}{m\,(ax^{m-1} + x^m)^{\frac{m-1}{m}}}, \text{ and therefore}$$

$$y' = (ax^{m-1} + x^m)^{\frac{1}{m}} + \frac{(m-1)\,ax^{m-2} + mx^{m-1}}{m\,(ax^{m-1} + x^m)^{\frac{m-1}{m}}}(x' - x)$$

$$= \frac{ax^{m-1} + (m-1)\,ax^{m-2}x' + mx^{m-1}x'}{mx^{m-1}\left(\dfrac{a}{x} + 1\right)^{\frac{m-1}{m}}}$$

$$= \frac{a + (m-1)\,a\dfrac{x'}{x} + mx'}{m\left(\dfrac{a}{x}+1\right)^{\frac{m-1}{m}}},$$

which is the equation to the tangent:

now, if $x = \infty$, $y = \infty$, and the equation to the asymptote is

$$y' = \frac{a + mx'}{m}, \text{ or } y' - x' = \frac{a}{m}.$$

If m be less than 1, when $x = 0$, $y = \infty$, and the equation to the asymptote becomes $y' = \infty$, or another asymptote in this case passes through the origin of the co-ordinates, and is perpendicular to the axis of x.

If $m = 2$, we have $y' = x' + \frac{a}{2}$, which is the equation to the asymptote of the rectangular hyperbola.

153. The equations to the asymptotes of curves may also be determined without the aid of the Differential Calculus, as will readily be seen by the following examples.

Ex. 1. In the common hyperbola $y = \frac{b}{a}\sqrt{x^2+2ax}$: whence expanding $\sqrt{x^2+2ax}$ by the binomial theorem, we find

$$y = \frac{b}{a}\left\{x + a - \frac{a^2}{2x} + \frac{a^3}{2x^2} - \&c.\right\},$$

wherefore if $x = \infty$, we shall have $y = \frac{b}{a}(x+a)$, which is the equation to the infinite part of the curve, and therefore to the rectilineal asymptote, since it is of only one dimension.

Ex. 2. Let $x^3 - 3axy + y^3 = 0$; and suppose the equation of a rectilineal asymptote to be $y = \alpha x + \beta$; then, by substitution, we shall have

$$0 = (\alpha^3 + 1)\, x^3 + 3\alpha\, (\alpha\beta - a)\, x^2 + \&c.;$$

whence $\alpha^3 + 1 = 0$, or $\alpha = -1$; $\alpha\beta - a = 0$, or $\beta = -a$, and the required asymptote is the straight line whose equation is

$$y = -x - a:$$

hence also, if $x = 0$, $y = -a$, and if $y = 0$, $x = -a$, agreeably to what is proved in the fifth example of article (150).

Ex. 3. Let $ax^4 + 4c^3xy - by^4 = 0$; then, if the equation to a rectilineal asymptote be $y = \alpha x + \beta$, we shall, as before, have

$$0 = (b\alpha^4 - a)\, x^4 + 4b\alpha^3\beta x^3 + \&c.;$$

from which we must have $b\alpha^4 - a = 0$, or $\alpha = \frac{a^{\frac{1}{4}}}{b^{\frac{1}{4}}}$ and $\beta = 0$, so that the equation to the required asymptote is $y = \frac{a^{\frac{1}{4}}}{b^{\frac{1}{4}}}\, x$, as in the sixth example of the same article.

It is evident that in order to determine all the rectilineal asymptotes by this method, we must find all the series by which y can be expressed in descending powers of x.

154. In the examples of the last article, the method pursued has enabled us to determine the equation of the *rectilineal* asymptote of a curve; but it is obvious that the equations to *curvilineal* asymptotes may be determined in the same manner.

For, if by solving the equation of the curve with respect to y, we find

$$y = ax^n + bx^{n-1} + \&c. + gx + h + \frac{k}{x} + \frac{l}{x^2} + \&c.,$$

and then suppose $x=\infty$, the equation to the curve at an infinite distance becomes

$$y=ax^n+bx^{n-1}+\&c.+gx+h,$$

which is also the equation to an asymptotic curve of n dimensions.

If $n=1$, $y=ax+b$, which is the equation to a rectilineal asymptote as before.

If $n=2$, $y=ax^2+bx+c$, which is the equation to a common parabolic asymptote.

If n be greater than 2, the asymptote is a parabola of the n^{th} order.

Also, when $n=1$, the infinite arc, whose equation is

$$y=ax+b+\frac{k}{x},$$

is hyperbolic, and approaches infinitely nearer to the proposed curve than the rectilineal asymptote whose equation is

$$y=ax+b.$$

Ex. Let the equation of the curve proposed be

$$ay^3-x^3y-ax^3=0;$$

then, by *Maclaurin's* Theorem, we have as in article (71),

$$y=x+\frac{x^2}{3a}-\frac{x^4}{3^4a^3}+\frac{x^8}{3^5a^4}-\&c.:$$

$$y=-a-\frac{a^4}{x^3}-\frac{3a^7}{x^6}-\frac{12a^{10}}{x^9}-\&c.:$$

$$y=\pm\frac{x^{\frac{3}{2}}}{a^{\frac{1}{2}}}+\frac{a}{2}\mp\frac{3a^{\frac{5}{2}}}{8x^{\frac{3}{2}}}+\&c.:$$

the first of these equations shews that there is no asymptote of any kind cutting the axis of x at a finite distance:

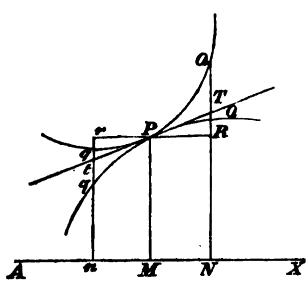
from the second we determine an asymptote parallel to the axis of x, whose equation is $y = -a$:

and the third discovers the existence of a parabolic asymptote whose equation is $2a^{\frac{1}{2}}y = \pm 2x^{\frac{3}{2}} + a^{\frac{3}{2}}$.

IV. DIRECTION OF CURVATURE.

155. *To find the position of a curve at any point with respect to its tangent.*

Let AP be any curve whose co-ordinates AM and MP at the point P are x and y respectively: take $MN = Mn = h$,



draw the ordinates NQ, nq and produce them, if necessary, to meet the tangent at P in T and t; also through P draw rPR parallel to the axis of x; then it has been seen that

$$NQ = y + \frac{dy}{dx}\frac{h}{1} + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \&c.;$$

whence we have immediately

$$RQ = \frac{dy}{dx}\frac{h}{1} + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \&c.$$

$$= PR \tan TPR + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \&c.$$

$$= TR + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \&c.;$$

$$\text{and } \therefore\ QR - TR = \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \&c.:$$

now, it is obvious that if the curve lie above its tangent at P or be convex to the axis of x, QR is greater than TR and therefore we must have

$$\frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \&c. \text{ positive};$$

but if the curve lie below its tangent at P or be concave to the same axis, QR is manifestly less than TR, and therefore

$$\frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \&c. \text{ is negative}:$$

also, since by the continual diminution of h, the first term $\frac{d^2y}{dx^2}\frac{h^2}{1.2}$ will become greater than the sum of all the succeeding terms of the series, it follows that the algebraical sign of this series will then be the same as that of its first term; and therefore according as a curve is convex or concave to the line of abscissas, the second differential coefficient $\frac{d^2y}{dx^2}$ of the ordinate will be positive or negative. The same is easily proved of the part of the curve on the other side of P.

Hence, conversely, a curve will be convex or concave to the line of abscissas, according as the second differential coefficient of the ordinate is positive or negative.

In what has just been said, the ordinate y was supposed to be positive; but if it were negative, the contrary results would be obtained, and we may conclude more generally, that a curve will be convex or concave to the line of abscissas according as the ordinate and its second differential coefficient have the same or different algebraical signs.

156. Cor. 1. Since, by the last article, $QT = QR - TR$

$$= \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \&c.,$$

we shall have

$$\frac{QT}{PR^2} = \frac{\frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \&c.}{h^2}$$

$$= \frac{d^2y}{dx^2}\frac{1}{1.2} + \frac{d^3y}{dx^3}\frac{h}{1.2.3} + \&c.;$$

and thence the ultimate value of $\frac{QT}{PR^2}$ or of $\frac{QT}{MN^2} = \frac{d^2y}{1.2.dx^2}$.

157. Cor. 2. Since by construction $MN = Mn$, we shall have, when h is diminished *sine limite*, the arc PQ equal to the arc Pq: also, if the differential coefficients be denoted by p, q, r, &c., we have seen that

$$QT = q\frac{h^2}{1.2} + r\frac{h^3}{1.2.3} + \&c.$$

$$qt = q\frac{h^2}{1.2} - r\frac{h^3}{1.2.3} + \&c.$$

whence we immediately obtain

$$\frac{QT}{qt} = \frac{q\frac{h^2}{1.2} + r\frac{h^3}{1.2.3} + \&c.}{q\frac{h^2}{1.2} - r\frac{h^3}{1.2.3} + \&c.} = \frac{q\frac{1}{1.2} + r\frac{h}{1.2.3} + \&c.}{q\frac{1}{1.2} - r\frac{h}{1.2.3} + \&c.},$$

the ultimate value of which is obviously 1: that is, if the points Q and q be joined and be supposed to move up continually towards P, the chord Qq ultimately becomes parallel to the tangent TPt at the middle point of the curve.

Again, if the chord PQ be drawn, we shall have

$$\frac{\sin QPT}{\sin PQT} = \frac{QT}{PT},$$

which is obviously less than

$$\frac{q\dfrac{h^2}{1.2} + r\dfrac{h^3}{1.2.3} + \&c.}{h} \text{ or } q\frac{h}{1.2} + r\frac{h^2}{1.2.3} + \&c.,$$

and therefore vanishes when h is diminished *sine limite:* whence the angle PQT being finite, it follows that the angle QPT contained between the chord and tangent ultimately vanishes.

158. Cor. 3. In the last article it has been shewn that that the angle QPT ultimately vanishes: and from this we immediately conclude that the arc PQ, its chord PQ and its tangent PT are ultimately coincident, and therefore equal to one another.

Also, in the last article but one, it has been proved that when h is indefinitely diminished, the value of

$$\frac{QT}{MN^2} \text{ or } \frac{QT}{PR^2} = \tfrac{1}{2}q:$$

wherefore, the values of the differential coefficients p, q, r, &c. being constant magnitudes for any assigned point in the curve, it follows that

$$QT \propto PR^2 \propto PR^2(1+p^2) \propto PT^2:$$

that is, the subtense QT ultimately varies as the square of the tangent PT, and therefore as the square of the corresponding chord or arc PQ.

In these corollaries, it scarcely need be observed that the differential coefficients p, q, r, &c. have been supposed to be finite magnitudes.

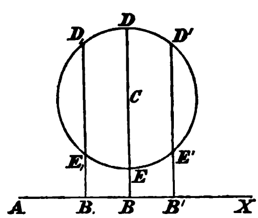
Ex. 1. Let the curve be a circle whose equation is

$a^2=(x-b)^2+(y-c)^2$; then is $y=c\pm\sqrt{a^2-(x-b)^2}$;

$$\therefore \frac{dy}{dx}=\mp\frac{x-b}{\sqrt{a^2-(x-b)^2}} \text{ and } \frac{d^2y}{dx^2}=\mp\frac{a^2}{\{a^2-(x-b)^2\}^{\frac{3}{2}}};$$

wherefore if the upper sign be used in the value of y, the corresponding part of the curve is concave to the axis of x, but if the lower, it is convex.

This will readily appear from an examination of the



diagram, in which the upper part of the figure is concave to the axis of x and the lower part convex.

Ex. 2. Let the curve be the cubical parabola whose equation is $y^3=a^2x$; then we have immediately,

$$y=a^{\frac{2}{3}}x^{\frac{1}{3}},\ \frac{dy}{dx}=\frac{1}{3}\frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}} \text{ and } \frac{d^2y}{dx^2}=-\frac{2}{9}\frac{a^{\frac{2}{3}}}{x^{\frac{5}{3}}}:$$

now if x be positive, y is positive and $\frac{d^2y}{dx^2}$ negative, whence the curve is concave towards the axis of x; but if x be negative, y is negative and $\frac{d^2y}{dx^2}$ positive, and therefore the curve has still its concavity turned towards the same axis.

V. ORDERS OF CONTACT.

159. Def. If two curves whose equations are $y = f(x)$ and $y' = F(x')$ be referred to the same rectangular axes, and h be a common increment or decrement of their abscissas, we shall, from what has been previously said, have

$$y \pm \Delta y = f(x \pm h)$$
$$= y \pm \frac{dy}{dx}\frac{h}{1} + \frac{d^2y}{dx^2}\frac{h^2}{1.2} \pm \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \&c.$$

and

$$y' \pm \Delta y' = F(x' \pm h)$$
$$= y' \pm \frac{dy'}{dx'}\frac{h}{1} + \frac{d^2y'}{dx'^2}\frac{h^2}{1.2} \pm \frac{d^3y'}{dx'^3}\frac{h^3}{1.2.3} + \&c.:$$

then, on the supposition that x' is made equal to x;

if $y' = y$, the curves have a common point of intersection:

if $y' = y$ and $\frac{dy'}{dx'} = \frac{dy}{dx}$, the curves have a common tangent and a contact of the *first* order:

if $y' = y$, $\frac{dy'}{dx'} = \frac{dy}{dx}$ and $\frac{d^2y'}{dx'^2} = \frac{d^2y}{dx^2}$, the curves have a common tangent and a contact of the *second* order:

if $y' = y$, $\frac{dy'}{dx'} = \frac{dy}{dx}$, $\frac{d^2y'}{dx'^2} = \frac{d^2y}{dx^2}$ and $\frac{d^3y'}{dx'^3} = \frac{d^3y}{dx^3}$, the curves have a common tangent and a contact of the *third* order:

and so on: and generally,

if $y' = y$, $\frac{dy'}{dx'} = \frac{dy}{dx}$, $\frac{d^2y'}{dx'^2} = \frac{d^2y}{dx^2}$, &c., $\frac{d^ny'}{dx'^n} = \frac{d^ny}{dx^n}$, the curves have a common tangent and a contact of the nth order.

Hence, if all the differential coefficients be equal, the curves entirely coincide.

160. Cor. 1. The distance between the curves measured in the direction of the ordinate when the contact is of the first order, is infinitely greater than when it is of the second: when the contact is of the second order, the distance is infinitely greater than when it is of the third: and so on of the superior orders.

For, at a common point, we have $y' = y$, and therefore

$$\Delta y - \Delta y' = \left(\frac{dy}{dx} - \frac{dy'}{dx'}\right)\frac{h}{1} + \left(\frac{d^2y}{dx^2} - \frac{d^2y'}{dx'^2}\right)\frac{h^2}{1.2}$$

$$+ \left(\frac{d^3y}{dx^3} - \frac{d^3y'}{dx'^3}\right)\frac{h^3}{1.2.3} + \left(\frac{d^4y}{dx^4} - \frac{d^4y'}{dx'^4}\right)\frac{h^4}{1.2.3.4} + \&c.$$

$$+ \left(\frac{d^{n-1}y}{dx^{n-1}} - \frac{d^{n-1}y'}{dx'^{n-1}}\right)\frac{h^{n-1}}{1.2.3.\&c.(n-1)} + \left(\frac{d^ny}{dx^n} - \frac{d^ny'}{dx'^n}\right)\frac{h^n}{1.2.3.\&c.n}$$

$+ \&c.$:

let δ_0 = the distance when $y' = y$;

$\delta_1 = \ldots\ldots\ldots\ldots\ldots\ldots \; y' = y$ and $\dfrac{dy'}{dx'} = \dfrac{dy}{dx}$;

$\delta_2 = \ldots\ldots\ldots\ldots\ldots\ldots \; y' = y, \; \dfrac{dy'}{dx'} = \dfrac{dy}{dx}$ and $\dfrac{d^2y'}{dx'^2} = \dfrac{d^2y}{dx^2}$;

&c...

and let the successive coefficients of h in the value of $\Delta y - \Delta y'$ be represented by P, Q, R, S, &c.; then we shall have

$$\frac{\delta_0}{\delta_1} = \frac{Ph + Qh^2 + Rh^3 + \&c.}{Qh^2 + Rh^3 + Sh^4 + \&c.} = \infty, \text{ when } h = 0;$$

$$\frac{\delta_1}{\delta_2} = \frac{Qh^2 + Rh^3 + Sh^4 + \&c.}{Rh^3 + Sh^4 + Th^5 + \&c.} = \infty, \text{ when } h = 0;$$

&c...

$$\frac{\delta_n}{\delta_{n+1}} = \frac{Vh^{n-1} + Wh^n + Xh^{n+1} + \&c.}{Wh^n + Xh^{n+1} + Yh^{n+2} + \&c.} = \infty, \text{ when } h = 0:$$

that is, δ_0, δ_1, δ_2, &c., δ_n are infinitely greater than δ_1, δ_2, δ_3, &c., δ_{n+1} respectively.

161. Cor. 2. Since each of the quantities δ_0, δ_1, δ_2, &c., δ_n, δ_{n+1} is indefinitely greater than any one of those that succeed it, it follows that if two curves have contact of any order whatever, no other curve which has an inferior order of contact with either of them, can be drawn so as to pass between them.

162. *To determine the nature and conditions of the contact subsisting between a straight line and any proposed curve.*

Let $y=f(x)$ be the equation of the proposed curve, $y'=ax'+\beta$ that of the straight line: then we have

$$\frac{dy'}{dx'}=a, \quad \frac{d^2y'}{dx'^2}=0, \text{ \&c.}:$$

whence at a point of contact of the first order we must have

$$x'=x, \quad y'=y, \quad \frac{dy'}{dx'}=\frac{dy}{dx};$$

or denoting the successive differential coefficients by p, q, r, &c., p', q', r', &c. we must have

$$x'=x, \quad y'=y, \quad p'=p:$$

$$\text{whence } a=p \text{ and } \beta=y-px,$$

and the equation $y'=ax'+\beta$ becomes

$$y'-y=p(x'-x):$$

in other words, since the value of x which satisfies the equations $y'=y$ and $p'=p$ does not necessarily satisfy the equations $q=0$, $r=0$, &c. the contact of a straight line with a proposed curve is *generally* of the first order, and the line itself is the rectilineal tangent to the curve, as appears also from (140), and may be constructed as before shewn.

It is moreover manifest that no other straight line can be drawn through the point of contact between this line and the

curve; for if there could, a different value of p would be required to satisfy the characterizing equations. The line thus determined is therefore called the *Osculating* straight line, and if the second, third, &c. differential coefficients corresponding to the point of concourse vanish, the *Osculation* will be of the second, third, &c. orders, a circumstance entirely dependent upon the peculiarity of the case.

163. *To investigate the nature and circumstances of the contact between a circle and a given curve.*

$$\text{Let } y = f(x) \text{ and } (x' - \alpha)^2 + (y' - \beta)^2 = \gamma^2,$$

be the equations of the proposed curve and circle respectively; then using the notation adopted in the last article for the successive differential coefficients, we must have at the point of concourse,

$$x' = x, \quad y' = y, \quad p' = p, \quad q' = q, \quad \&c.$$

but by successive differentiation we obtain

$$x' - \alpha + (y' - \beta)p' = 0, \quad 1 + (y' - \beta)q' + p'^2 = 0;$$

whence we have immediately

$$x - \alpha + (y - \beta)p = 0 \text{ and } 1 + (y - \beta)q + p^2 = 0:$$

from which together with the equation

$$(x - \alpha)^2 + (y - \beta)^2 = \gamma^2,$$

the three magnitudes α, β and γ may be exhibited in terms of x, y, p and q: thus

$$(x - \alpha)q + (y - \beta)pq = 0,$$

$$p + (y - \beta)pq + p^3 = 0:$$

$$\therefore \ (x - \alpha)q = p(1 + p^2) \text{ and } x - \alpha = \frac{p(1 + p^2)}{q},$$

$$\text{whence } \alpha = x - \frac{p(1 + p^2)}{q}:$$

$$\text{also, } (y-\beta)q = -(1+p^2) \text{ and } y-\beta = -\frac{(1+p^2)}{q},$$

$$\text{whence } \beta = y + \frac{1+p^2}{q}:$$

$$\text{and therefore } \gamma^2 = (x-\alpha)^2 + (y-\beta)^2$$

$$= \frac{p^2(1+p^2)^2}{q^2} + \frac{(1+p^2)^2}{q^2} = \frac{(1+p^2)^3}{q^2};$$

$$\text{or } \gamma = \frac{(1+p^2)^{\frac{3}{2}}}{q}:$$

which will manifestly be positive or negative according as q is positive or negative, and therefore by (155), according as the curve is convex or concave towards the axis of x; and the co-ordinates of the centre and the magnitude of the radius, of the circle, are thus exhibited in terms of the co-ordinates of the proposed curve: whence, if the point in the curve be assigned, the circle having with it, at that point, contact of the second order, is completely determined.

Also, to determine the three quantities α, β and γ, it is manifest that the three equations above given are necessary and sufficient, and therefore these values so determined, need not necessarily satisfy any more of the conditions expressed by $r'=r$, $s'=s$, &c.; or, in other words, the order of contact between the curve and circle, need not be higher than the second, though in particular cases it may be so, as explained in the last article.

Since then it is obvious that no other circle can in general be made to pass between the curve and the circle above defined, it is customary, as in the last article, to designate it the *Osculating Circle* at the point (x, y) of the curve proposed, the osculation being of the second order, except in the cases above alluded to.

164. Cor. In the investigation above given, we have seen that

$$(x-\alpha) + (y-\beta)p = 0:$$

and this put in a different form, becomes

$$\beta - y = -\frac{dx}{dy}(a - x),$$

which, as appears from (148), is the equation of the normal to the point (x, y) of the proposed curve: therefore, the co-ordinates a, β of the centre of the osculating circle, are also the co-ordinates of a point in the normal, or the diameter of the osculating circle passing through the point of contact is a normal to the curve.

165. *To find the nature and circumstances of the contact subsisting between a proposed curve and a parabolic curve of any order.*

Let $y = f(x)$ be the equation of the curve proposed, and

$$y' = a + \beta x' + \gamma x'^2 + \delta x'^3 + \&c. + \lambda x'^m,$$

that of a parabolic curve of the m^{th} order: then we shall have

$$\frac{dy'}{dx'} = \beta + 2\gamma x' + 3\delta x'^2 + \&c. + m\lambda x'^{m-1};$$

$$\frac{d^2 y'}{dx'^2} = 1.2\gamma + 2.3\delta x' + \&c. + (m-1)m\lambda x'^{m-2};$$

$$\frac{d^3 y'}{dx'^3} = 1.2.3\delta + \&c. + (m-2)(m-1)m\lambda x'^{m-3};$$

&c. ....

$$\frac{d^m y'}{dx'^m} = 1.2.3.\&c.(m-2)(m-1)m\lambda:$$

whence, in addition to the conditions expressed by $x' = x$ and $y' = y$, we may obviously have the m following equations:

$$p' = p, \; q' = q, \; r' = r, \; \&c., \; t' = t:$$

which will therefore designate a contact of the m^{th} order, and it is evident that by means of these and the equation $y' = y$, the $m + 1$ magnitudes, a, β, γ, &c. λ may be expressed in terms of the co-ordinates and successive differential coefficients.

Also, the order of osculation cannot in general be greater than the m^{th}, since the values thus determined need not satisfy any more of the equations

$$\frac{d^{m+1}y}{dx^{m+1}}=0,\ \frac{d^{m+2}y}{dx^{m+2}}=0,\ \&c.$$

Ex. Taking the conical parabola, or the parabola of the second order, we have

$$y'=\alpha+\beta x'+\gamma x'^2:$$

$$\therefore\ \frac{dy'}{dx'}=\beta+2\gamma x',\ \frac{d^2y'}{dx'^2}=2\gamma,\ \frac{d^3y'}{dx'^3}=0,\ \&c.$$

whence as before, we shall find the equations characterizing the circumstances of the contact, to be

$$y=\alpha+\beta x+\gamma x^2,\ p=\beta+2\gamma x \text{ and } q=2\gamma,$$
$$\text{which give } \gamma=\tfrac{1}{2}q,\ \beta=p-qx,$$
$$\text{and } \alpha=y-px+\tfrac{1}{2}qx^2:$$

wherefore, corresponding to any assigned values of x and y, or to any given point in the curve, the parabolic curve whose equation is

$$y'=(y-px+\tfrac{1}{2}qx^2)+(p-qx)x'+\tfrac{1}{2}qx'^2,$$

is completely determined: that is,

$$\text{since } y'=\alpha+\beta x'+\gamma x'^2=\gamma\left(\frac{\alpha}{\gamma}+\frac{\beta}{\gamma}x'+x'^2\right)$$

$$=\gamma\left\{\frac{\alpha}{\gamma}+\left(x'+\frac{\beta}{2\gamma}\right)^2-\frac{\beta^2}{4\gamma^2}\right\}=\gamma\left(x'+\frac{\beta}{2\gamma}\right)^2+\frac{4\alpha\gamma-\beta^2}{4\gamma},$$

$$\text{and therefore } y'-\frac{4\alpha\gamma-\beta^2}{4\gamma}=\gamma\left(x'+\frac{\beta}{2\gamma}\right)^2,$$

we shall have the co-ordinates of the vertex of the osculating parabola whose axis is perpendicular to the axis of x, equal to $-\dfrac{\beta}{2\gamma}$ and $\dfrac{4\alpha\gamma-\beta^2}{4\gamma}$ respectively, and the latus rectum equal to

$\frac{1}{\gamma}$, which may be replaced by their values above found, p and q being determined from the equation to the curve proposed.

Here it may be observed, that an infinite number of parabolas of the second order may have with the curve proposed, contact of the *first* order, since the equations $y'=y$ and $p'=p$ will enable us to exhibit in terms of x, y and p only two of the magnitudes a, β and γ, belonging to the equation

$$y'=a+\beta x'+\gamma x'^2;$$

but that the one above-determined alone has in general, contact of the *second* order with it, and is therefore designated the *Osculating* parabola of the second order at the point of concourse. Similarly of parabolic curves of higher orders: and the determination of the magnitudes a, β, γ, &c. will generally be greatly facilitated by the removal of the origin of the co-ordinates to the point under immediate consideration.

166. Cor. The $m+1$ equations characterizing intersection and successive orders of contact, namely,

$$y'=y,\ p'=p,\ q'=q,\ \&c.,\ t'=t,$$

may, as has been before observed, be the means of exhibiting in terms of the co-ordinates and their differential co-efficients, the values of $m+1$ constants in the equation to any curve $y'=F(x')$; and since this series of equations indicates in general the existence of contact of the m^{th} order, it follows as before, that *Osculating* curves of any other given species whose equations involve not more than $m+1$ constant quantities, may be similarly determined: but if the equation $y'=F(x')$ be supposed to contain more than $m+1$ constants, since only $m+1$ of them can be thus determined, it is manifest that there may be an infinite number of curves of a given species, which may have contact of the m^{th} order with the curve proposed, not one of which has osculation with it, because its nature and circumstances cannot be completely defined.

167. *To ascertain generally the nature and circumstances of the contact subsisting between any proposed curve, and a curve of the second order.*

Let $y=f(x)$ denote the proposed curve, and

$$y'^2+(\alpha x'+\beta)y'+\gamma+\delta x'+\epsilon x'^2=0,$$

any curve of the second order: then it is obvious that the characterizing equations

$$y'=y,\ p'=p,\ q'=q,\ r'=r \text{ and } s'=s,$$

will enable us to exhibit the five quantities α, β, γ, δ and ϵ, in terms of the co-ordinates and their differential coefficients, and will indicate generally the existence of contact and osculation of the fourth order.

Hence, there may obviously exist an infinite number of curves of the second order having, with the curve proposed, contact of the *third* order, but only the one determined as above, which *osculates* the curve, and has contact of the *fourth* order with it.

Also, since by solving the equation with respect to y', we find

$$y'=\frac{-\alpha x'-\beta\pm\sqrt{(\alpha^2-4\epsilon)x'^2+(2\alpha\beta-4\delta)x'+\beta^2-4\gamma}}{2},$$

it is obvious that the nature of the osculating curve to the proposed curve at any given point, will depend entirely upon the value of $\alpha^2-4\epsilon$, and will be an ellipse, parabola or hyperbola, according as α^2 is less than, equal to, or greater than 4ϵ, the magnitudes of α and ϵ being first exhibited in terms of the co-ordinates of the curve proposed.

168. COR. 1. By assuming $\alpha^2-4\epsilon=0$, we manifestly obtain one or more points in the curve proposed, at which the osculating curve of the second order is a parabola, the order of contact being the fourth: and since at these points the value of $\alpha^2-4\epsilon$ passes through zero and changes its algebraical sign, it follows that on one side of the points thus determined, the osculating curves of the second order are ellipses, and on the other hyperbolas, whose magnitudes and positions may therefore be ascertained.

169. Cor. 2. Hence, also, *geometrical* representations of the successive differential coefficients of the ordinate may easily be had by drawing through any proposed point P of a curve, the rectilineal tangent and osculating parabolas of successive orders; thus, if NQ_1, NQ_2, NQ_3, &c. denote the ordinates of the rectilineal tangent and osculating parabolas of the second, third, &c. orders corresponding to the abscissa $x+h$, and PQ_0 be drawn parallel to the axis of x, we shall obviously have

$$NQ_0 = y,\ NQ_1 = y + \frac{dy}{dx}h;$$

$$NQ_2 = y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1.2};$$

$$NQ_3 = y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{1.2.3}, \text{ \&c.};$$

whence will be immediately obtained

$$Q_0Q_1 = \frac{dy}{dx}h,\ Q_1Q_2 = \frac{d^2y}{dx^2}\frac{h^2}{1.2},\ Q_2Q_3 = \frac{d^3y}{dx^3}\frac{h^3}{1.2.3}, \text{ \&c.}$$

which, if h be supposed equal to the indeterminate magnitude dx, give

$$Q_0Q_1 = dy,\ 1.2\ Q_1Q_2 = d^2y,\ 1.2.3\ Q_2Q_3 = d^3y, \text{ \&c.}$$

170. *If the contact between two curves be of an even order, they have only contact at the point of concourse, but if it be of an odd order, they have both contact and intersection.*

For, let $y = f(x)$ and $y' = F(x')$ be the equations to the two curves, and h a common increment or decrement of their abscissæ; then we shall have immediately

$$y_1 = f(x+h) = y + p\frac{h}{1} + q\frac{h^2}{1.2} + r\frac{h^3}{1.2.3} + \text{\&c.}$$

$$y'_1 = F(x'+h) = y' + p'\frac{h}{1} + q'\frac{h^2}{1.2} + r'\frac{h^3}{1.2.3} + \text{\&c.}$$

also,

$$y_{-1}=f(x-h)=y-p\frac{h}{1}+q\frac{h^2}{1.2}-r\frac{h^3}{1.2.3}+\&c.$$

$$y'_{-1}=F(x'-h)=y'-p'\frac{h}{1}+q'\frac{h^2}{1.2}-r'\frac{h^3}{1.2.3}+\&c.:$$

hence, if δ_n and δ_{-n} denote the differences of the ordinates of the two curves corresponding to the common abscissæ $x+h$ and $x-h$ when the contact subsisting between them is of the n^{th} order, we shall manifestly have

$$\delta_n=\left\{\frac{d^{n+1}y}{dx^{n+1}}-\frac{d^{n+1}y'}{dx'^{n+1}}\right\}\frac{h^{n+1}}{1.2.3.\&c.\ (n+1)}+\&c.$$

$$\delta_{-n}=\pm\left\{\frac{d^{n+1}y}{dx^{n+1}}-\frac{d^{n+1}y'}{dx'^{n+1}}\right\}\frac{h^{n+1}}{1.2.3.\&c.\ (n+1)}\mp\&c.:$$

in the latter of which it is obvious that the upper or lower sign must be used according as n is odd or even:

wherefore if h be taken so small that the signs of the sums of these series may be the same as the signs of their first terms, the values of δ_n and δ_{-n} will have the same or different signs according as n is odd or even; and therefore on both sides of the point of concourse, the latter curve will lie below or above the former if the contact be of an odd order; and on one side of the said point it will lie below it and on the other above it, when the contact is of an even order: that is, in the former case there is contact only, and in the latter both contact and intersection.

171. Cor. Since contact of the first and second orders generally subsists between a curve and its rectilineal tangent and osculating circle respectively, it follows that the rectilineal tangent only touches the curve, and that the osculating circle in general both touches and cuts it at the point of concourse.

172. The conclusions at which we have arrived in the preceding pages, are sometimes obtained from other considerations connected with the *geometrical* rather than the *analytical* view of the subject: thus, if a curve be first supposed to

intersect another in m points, and its constants be expressed in terms of the co-ordinates of the latter, and the m points be then supposed to be united in one, the results may be defined to belong to the different orders of osculation.

VI. CIRCLE OF CURVATURE.

173. We have just seen that the order of osculation subsisting between two curves is entirely dependent upon the number of constants involved in their equations: and consequently if with a proposed curve another be determined having osculation of any proposed order, it will always be possible to ascertain one of another species which shall have with it osculation of a still higher order: in other words, there exists no limit to the orders of contact which may subsist between a proposed curve and curves of other species. In a great variety of physical problems to which the *Differential Calculus* is usually applied and wherein the properties of curves are concerned, it happens to be unnecessary to have recourse to differential coefficients of the co-ordinates of higher orders than the second: and it therefore obviously follows that the osculating circle will on this account acquire a degree of importance which it would otherwise have had no claim to, inasmuch as its co-ordinates and their first and second differential coefficients are respectively equal to those of the curve to which it belongs. In such problems, especially when considered in a *geometrical* point of view, the circle, from the circumstance of its admitting of all degrees of magnitude by the variation of its radius, from the facility of its description and the simplicity of its form and properties, is preferred to all other curves wherein the same degree of osculation may take place. The osculating circle having therefore its curvature more nearly equal to that of the curve proposed than any other circle whatever, is, in such cases as those above referred to, designated *The Circle of Curvature:* and the radius, diameter, chords, &c. of this circle are styled *The Radius, Diameter, Chords, &c. of Curvature* of the proposed curve at the point where the contact takes place.

174. COR. 1. Hence if k and k' represent the parts of the ordinate of a curve and the above-mentioned circle corresponding to the common abscissa $x+h$, and intercepted between them and their common tangent, we shall have

$$\frac{k}{k'} = \frac{q\dfrac{h^2}{1.2} + r\dfrac{h^3}{1.2.3} + \&c.}{q'\dfrac{h^2}{1.2} + r'\dfrac{h^3}{1.2.3} + \&c.},$$

as appears from (156): whence the limit of $\dfrac{k}{k'}$ = the limit of

$$\frac{\dfrac{1}{1.2}q + \dfrac{1}{1.2.3}rh + \&c.}{\dfrac{1}{1.2}q' + \dfrac{1}{1.2.3}r'h + \&c.} = \frac{q}{q'} = 1:$$

and the same will manifestly be true of any two lines drawn equally inclined to the common ordinate from the points of intersection: and this in fact, includes the *geometrical* definition of the circle of curvature.

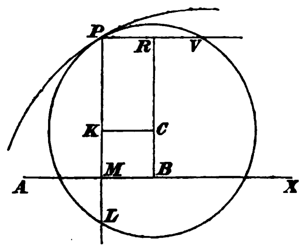
175. COR. 2. On the supposition, therefore, that the curvature of the osculating circle is equal to that of the curve, we conclude that the curvature of any curve varying as $\dfrac{1}{\gamma}$ varies as $\dfrac{q}{(1+p^2)^{\frac{3}{2}}}$ varies as q, (since the curves have necessarily a common tangent and therefore $(1+p^2)^{\frac{3}{2}}$ is constant,) $\propto \dfrac{QT}{PR^2} \propto \dfrac{QT}{PQ^2}$ from (158), $\propto$ limit of $\dfrac{\text{subtense}}{(\text{arc})^2}$.

176. *To find expressions for determining the radius of curvature of any proposed curve, and also for the chords of curvature parallel to the co-ordinate axes.*

If α and β be the co-ordinates of the centre, and γ the radius of the osculating circle, it has been proved that

$$\gamma = \frac{(1+p^2)^{\frac{3}{2}}}{q}, \quad \alpha = x - \frac{p(1+p^2)}{q} \text{ and } \beta = y + \frac{1+p^2}{q}:$$

and from these the radius and the co-ordinates of the centre of curvature may immediately be expressed in terms of the co-ordinates of the curve:



also, the chord of curvature parallel to the axis of $x = PV$

$$= 2PR = 2MB = 2(AB - AM)$$

$$= 2(\alpha - x) = -\frac{2p(1+p^2)}{q};$$

and the chord of curvature parallel to the axis of $y = PL$

$$= 2PK = 2CR = 2(MP - BC)$$

$$= 2(y - \beta) = \frac{2(1+p^2)}{q}.$$

If it be thought desirable that the radius of curvature, which is not generally measured in the direction of either of the co-ordinate axes, should always be positive, we have only to take $\gamma = \dfrac{(1+p^2)^{\frac{3}{2}}}{q}$ when the curve is convex to the axis of x, and $\gamma = \dfrac{(1+p^2)^{\frac{3}{2}}}{-q}$ when it is concave, as appears from (155).

177. Cor. If PF were any other chord of the circle passing through a point whose co-ordinates are a, b: and CP, CF were supposed to be drawn, as also a tangent PT at the point P, we should have

$$\frac{PF}{PC} = \text{chord } PCF = 2\sin\tfrac{1}{2}PCF = 2\sin\tfrac{1}{2}(\pi - 2CPF)$$

$$= 2\cos CPF = 2\sin TPF = 2\sin(MPF + TPM)$$

$$= 2\{\sin MPF\cos TPM + \cos MPF\sin TPM\}$$

$$= 2\cos MPF\cos TPM\{\tan MPF + \tan TPM\}$$

$$= \frac{2}{\sec MPF\sec TPM}\{\tan MPF + \tan TPM\}$$

$$= \frac{2}{\sqrt{1+\left(\frac{a-x}{y-b}\right)^2}\sqrt{1+\frac{1}{p^2}}}\left\{\frac{a-x}{y-b}+\frac{1}{p}\right\}$$

$$= \frac{2\{y-b-p(x-a)\}}{\sqrt{1+p^2}\sqrt{(y-b)^2+(x-a)^2}};$$

$$\text{whence } PF = \frac{(1+p^2)^{\frac{3}{2}}}{q}\;\frac{2\{y-b-p(x-a)\}}{\sqrt{1+p^2}\sqrt{(y-b)^2+(x-a)^2}}$$

$$= \frac{2(1+p^2)\{y-b-p(x-a)\}}{q\sqrt{(y-b)^2+(x-a)^2}}:$$

and if $a=0$, $b=0$, or the chord pass through the origin of the co-ordinates, we find

$$PF = \frac{2(1+p^2)(y-px)}{q\sqrt{x^2+y^2}}.$$

Ex. 1. The equation to a conic section whose *latus rectum* is m, being $y = \sqrt{mx+nx^2}$, we have

$$p = \frac{\frac{1}{2}m + nx}{\sqrt{mx+nx^2}} \text{ and } q = -\frac{m^2}{4(mx+nx^2)^{\frac{3}{2}}};$$

whence is immediately obtained

$$\gamma = \frac{(1+p^2)^{\frac{3}{2}}}{q} = \frac{\left\{1+\left(\frac{\frac{1}{2}m+nx}{\sqrt{mx+nx^2}}\right)^2\right\}^{\frac{3}{2}}}{-\frac{m^2}{4(mx+nx^2)^{\frac{3}{2}}}}$$

$$= -\frac{4}{m^2}\{\tfrac{1}{4}m^2 + (n+1)mx + (n+1)nx^2\}^{\frac{3}{2}}$$

$$= -\frac{4}{m^2}\{\tfrac{1}{4}m^2 + (n+1)y^2\}^{\frac{3}{2}},$$

which is the radius of curvature, the negative sign merely shewing that the curve is concave towards the axis of x:

$$\text{also, } \alpha = x - \frac{p(1+p^2)}{q}$$

$$= \frac{\frac{1}{2}m^3 + 3(n+1)m^2x + 6(n+1)mnx^2 + 4(n+1)n^2x^3}{m^2},$$

$$\text{and } \beta = y + \frac{1+p^2}{q} = -\frac{4(n+1)(mx+nx^2)^{\frac{3}{2}}}{m^2} = -\frac{4(n+1)}{m^2}y^3;$$

which are the co-ordinates of the centre of curvature: and thus the magnitude and position of the circle of curvature are completely determined.

If n be positive, the curve is an hyperbola and the radius and co-ordinates of the centre of the circle of curvature are the quantities above exhibited.

If $n = 0$, the curve becomes a parabola, and we have

$$\gamma = -\frac{4}{m^2}\{\tfrac{1}{4}m^2 + mx\}^{\frac{3}{2}} = -\frac{4}{m^2}\{\tfrac{1}{4}m^2 + y^2\}^{\frac{3}{2}},$$

$$\alpha = \tfrac{1}{2}m + 3x \text{ and } \beta = -\frac{4x^{\frac{3}{2}}}{m^{\frac{1}{2}}} = -\frac{4y^3}{m^2}.$$

If n be negative, the curve is an ellipse, and we have

$$\gamma = -\frac{4}{m^2}\{\tfrac{1}{4}m^2 - (n-1)mx + (n-1)nx^2\}^{\frac{3}{2}}$$

$$= -\frac{4}{m^2}\{\tfrac{1}{4}m^2 - (n-1)y^2\}^{\frac{3}{2}}:$$

$$\alpha = \frac{\frac{1}{2}m^3 - 3(n-1)m^2x + 6(n-1)mnx^2 - 4(n-1)n^2x^3}{m^2},$$

$$\text{and } \beta = \frac{4(n-1)(mx-nx^2)^{\frac{3}{2}}}{m^2} = \frac{4(n-1)y^3}{m^2}.$$

At the vertex of each of these curves where $x=0$, $y=0$, we have

$$\gamma = -\tfrac{1}{2}m = -\tfrac{1}{2} \text{ the } \textit{latus rectum}, \ \alpha = \tfrac{1}{2}m \text{ and } \beta = 0.$$

Also, since the square of the normal $=\frac{1}{4}m^2 + (n+1)y^2$,

$$\text{we shall evidently have } \gamma = -\frac{4}{m^2} \times (\text{normal})^3:$$

that is, the radius of curvature in all the Conic Sections

$$= \frac{(\text{normal})^3}{(\frac{1}{2}\textit{ latus rectum})^2}.$$

Ex. 2. Of an ellipse, the co-ordinates being measured from the centre, the equation is

$$y = \frac{b}{a}\sqrt{a^2-x^2}; \ \therefore p = -\frac{bx}{a\sqrt{a^2-x^2}} \text{ and } q = -\frac{ab}{(a^2-x^2)^{\frac{3}{2}}};$$

wherefore by substitution, we obtain

$$\gamma = -\frac{\{a^4-(a^2-b^2)x^2\}^{\frac{3}{2}}}{a^4 b} = -a^2b^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)^{\frac{3}{2}},$$

$$\alpha = \frac{a^2-b^2}{a^4}x^3 \text{ and } \beta = -\frac{a^2-b^2}{b^4}y^3,$$

which completely determine the circle of curvature.

$$\text{Hence } PV = -2x\left\{\frac{a^4-(a^2-b^2)x^2}{a^4}\right\} = -2b^2x\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right),$$

$$\text{and } PL = -2y\left\{\frac{b^4-(a^2-b^2)y^2}{b^4}\right\} = -2a^2y\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right).$$

Ex. 3. In the hyperbola between the asymptotes $xy=a^2$; whence $p = -\frac{a^2}{x^2}$ and $q = \frac{2a^2}{x^3}$; and therefore by substitution

are obtained $\gamma = \frac{1}{2a^2}\left\{x^2 + \frac{a^4}{x^2}\right\}^{\frac{3}{2}} = \frac{(x^2+y^2)^{\frac{3}{2}}}{2a^2}$, the positive sign shewing that the curve is convex towards the axis of x, and therefore that the circle lies above the curve:

$$a = \frac{a^4 + 3x^4}{2x^3} = \frac{3a^4 + y^4}{2a^2 y} \text{ and } \beta = \frac{a^4 + 3y^4}{2y^3} = \frac{3a^4 + x^4}{2a^2 x}:$$

$$\text{also, } PV = \frac{a^4 + x^4}{x^3} \text{ and } PL = \frac{a^4 + y^4}{y^3}.$$

Ex. 4. In the logarithmic curve $y = a^x$, $p = k a^x$ and $q = k^2 a^x$; whence we immediately obtain

$$\gamma = \frac{(1 + k^2 a^{2x})^{\frac{3}{2}}}{k^2 a^x}, \quad a = x - \frac{1 + k^2 a^{2x}}{k} \text{ and } \beta = \frac{1 + 2k^2 a^{2x}}{k^2 a^x}:$$

$$\text{also, } PV = -\frac{2(1 + k^2 a^{2x})}{k} \text{ and } PL = \frac{2(1 + k^2 a^{2x})}{k^2 a^x}.$$

Ex. 5. In the Trochoid whose ordinary equation is

$$y = ma \operatorname{vers}^{-1}\frac{x}{a} + \sqrt{2ax - x^2},$$

we have $p = \frac{(m+1)a - x}{\sqrt{2ax - x^2}}$ and $q = \frac{max - (m+1)a^2}{(2ax - x^2)^{\frac{3}{2}}}$:

whence are obtained $\gamma = -\frac{\{(m+1)^2 a^2 - 2max\}^{\frac{3}{2}}}{(m+1)\ a^2 - \ max}$;

$$a = \frac{(m+1)^3 a^2 - 3m(m+1)ax + mx^2}{(m+1)a - mx},$$

$$\text{and } \beta = ma \operatorname{vers}^{-1}\frac{x}{a} - \left\{\frac{m(m+1)a - mx}{(m+1)a - mx}\right\}\sqrt{2ax - x^2}:$$

$$\text{also, } PV = 2\left\{\frac{(m+1)^3 a^2 - (m+1)(3m+1)ax + 2mx^2}{(m+1)a - mx}\right\},$$

$$\text{and } PL = -2\left\{\frac{(m+1)^2 a - 2mx}{(m+1)\,a - \;mx}\right\}\sqrt{2ax - x^2}.$$

If $m = 1$, we have $y = a \text{ vers}^{-1}\frac{x}{a} + \sqrt{2ax - x^2}$, which is the equation to a common cycloid: whence are found

$$\gamma = -2\sqrt{4a^2 - 2ax} = 2QB:$$

$$\alpha = 4a - x = 2AB - AM,$$

$$\text{and } \beta = a \text{ vers}^{-1}\frac{x}{a} - \sqrt{2ax - x^2} = QP - QM:$$

also, $PV = 4(2a - x) = 4MB$ and $PL = -4\sqrt{2ax - x^2} = 4QM$: AB being supposed to be the axis of the cycloid, and Q the point in which the ordinate intersects the generating circle.

Ex. 6. In the catenary, one of whose equations is

$$x = \frac{a}{2}\left\{e^{\frac{y}{a}} + e^{-\frac{y}{a}}\right\},$$

$$\text{we have } p = \frac{2}{e^{\frac{y}{a}} - e^{-\frac{y}{a}}} \text{ and } q = -\frac{4}{a}\,\frac{e^{\frac{y}{a}} + e^{-\frac{y}{a}}}{\left\{e^{\frac{y}{a}} - e^{-\frac{y}{a}}\right\}^3}:$$

$$\text{whence } \gamma = -\frac{a}{4}\left\{e^{\frac{y}{a}} + e^{-\frac{y}{a}}\right\}^2 = -\frac{x^2}{a}:$$

$$\alpha = a\left\{e^{\frac{y}{a}} + e^{-\frac{y}{a}}\right\} = 2x \text{ and } \beta = y - \frac{a}{4}\left\{e^{\frac{2y}{a}} - e^{-\frac{2y}{a}}\right\};$$

$$\text{also, } PV = -x \text{ and } PL = -\frac{a}{2}\left\{e^{\frac{2y}{a}} - e^{-\frac{2y}{a}}\right\}.$$

When $y = 0$, we find the corresponding value of γ to be $-a$, which is the radius of curvature at the vertex of a common parabola, whose *latus rectum* is $2a$.

Hence a very small arc of the catenary near its vertex may without sensible error be treated as the corresponding arc of a common parabola.

Ex. 7. If $\frac{dy}{dx}$ or $p = \frac{x^2}{\sqrt{a^4 - x^4}}$, which is a differential equation to the elastic curve, we have $q = \frac{2a^4 x}{(a^4 - x^4)^{\frac{3}{2}}}$:

$$\therefore \gamma = \frac{a^2}{2x}, \quad \alpha = \frac{x}{2} \text{ and } \beta = y + \frac{\sqrt{a^4 - x^4}}{2x}:$$

$$\text{also, } PV = -x \text{ and } PL = \frac{\sqrt{a^4 - x^4}}{x}.$$

Here we have $\gamma x = \frac{a^2}{2}$; or the radius of curvature varies inversely as the corresponding abscissa.

Ex. 8. To find the circle of curvature belonging to any point of a curve whose tangent intercepted by the axis of x is always of a given magnitude.

$$\text{Here } PT = \sqrt{y^2 + \left(\frac{y\,dx}{dy}\right)^2} = a \text{ or } y^2\left(\frac{dy}{dx}\right)^2 + y^2 = a^2\left(\frac{dy}{dx}\right)^2;$$

$$\text{whence } p = -\frac{y}{\sqrt{a^2 - y^2}} \text{ and } q = \frac{a^2 y}{(a^2 - y^2)^2}:$$

$$\therefore \gamma = \frac{a\sqrt{a^2 - y^2}}{y}, \quad \alpha = x + \sqrt{a^2 - y^2} \text{ and } \beta = \frac{a^2}{y}:$$

$$\text{also, } PV = 2\sqrt{a^2 - y^2} \text{ and } PL = 2\left(\frac{a^2 - y^2}{y}\right).$$

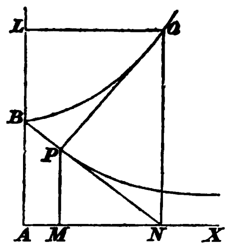
This curve is called the *Tractrix*, and is a species of *equitangential* curve, whose equation between the rectangular co-ordinates is

$$a + \sqrt{a^2 - y^2} = y e^{\frac{x + \sqrt{a^2 - y^2}}{a}},$$

from which also the conclusions above given may easily be derived.

Some curious properties appertain to the curve defined by this equation, one of which is the following.

Let P be a point in the curve referred to the rectangular co-ordinates $AM=x$, $MP=y$, and let PN be a tangent at P:



then if PQ and NQ be drawn respectively perpendicular to the tangent and axis, meeting in Q, the point Q will be the centre and the line QP the radius of the circle of curvature at P.

$$\text{For, } a=x+\sqrt{a^2-y^2}=AM+\sqrt{PN^2-PM^2}$$
$$=AM+MN=AN:$$

$$\text{and } \beta=\frac{a^2}{y}=\frac{PN^2}{PM}=NQ, \text{ by similar triangles.}$$

178. Cor. In the application of *Geometry* to curves, it is generally assumed that the limit of the intersection of two consecutive normals is the centre of the circle of curvature: but it is not difficult to prove independently, that this ultimate intersection coincides with the centre of curvature as above defined.

For, since the equation to the normal at a point of the curve whose co-ordinates are x and y, is by (148),

$$y_{\prime}-y=-\frac{1}{p}(x_{\prime}-x),$$

if we substitute α and β supposed to be the co-ordinates of the point of intersection, in the places of $x_{\prime}$ and $y_{\prime}$, we get

$$\beta - y = -\frac{1}{p}(\alpha - x), \text{ or } x - \alpha = -p(y - \beta);$$

therefore, since α and β remain the same for both points of the curve, we shall obtain, by differentiating with respect to the rest,

$$1 = q(\beta - y) - p^2:$$

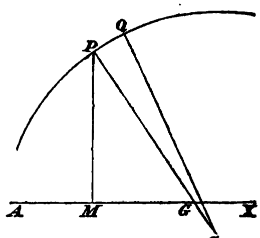
$$\text{whence } \beta - y = \frac{1 + p^2}{q} \text{ and } \alpha - x = -\frac{p(1 + p^2)}{q}:$$

$$\text{or } \alpha = x - \frac{p(1 + p^2)}{q} \text{ and } \beta = y + \frac{1 + p^2}{q},$$

which are the co-ordinates of the centre of curvature as before determined.

Without entering more fully into the principles of contact and osculation, the property just proved furnishes an easy method of determining the magnitude of the radius of the circle of curvature, whether x, y or s be supposed to be the independent variable, as will be evinced in the following articles.

179. First, if the normals PC and QC intersect in the



point C, the same notation being used as before and x being considered the principal variable, we shall have, when the arc PQ is indefinitely diminished,

$$\frac{PQ}{PC} = \angle PCQ, \text{ or } \frac{1}{\gamma} = \text{limit of } \frac{\angle PCQ}{PQ} = \frac{d(PGM)}{ds};$$

$$\therefore \gamma = \frac{ds}{d\tan^{-1}\left(\frac{dx}{dy}\right)} = \frac{ds}{d\cot^{-1}\left(\frac{dy}{dx}\right)}$$

$$= -\frac{dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\dfrac{d\left(\frac{dy}{dx}\right)}{1+\left(\frac{dy}{dx}\right)^2}} = -\frac{dx\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{d\left(\frac{dy}{dx}\right)}$$

$$= -\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \text{ or } = -\frac{\left(\frac{ds}{dx}\right)^3}{\frac{d^2y}{dx^2}};$$

and this differs from the value of γ found in (163) only in its algebraical sign, because the line PC measured from the curve *towards* the axis of x has here been supposed positive.

180. Secondly, if y be considered the independent variable, we have, by what has been done above,

$$\gamma = \frac{ds}{d\tan^{-1}\left(\frac{dx}{dy}\right)} = \frac{\frac{ds}{dy}\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}{\frac{d^2x}{dy^2}}$$

$$= \frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2x}{dy^2}} \text{ or } = \frac{\left(\frac{ds}{dy}\right)^3}{\frac{d^2x}{dy^2}}.$$

Ex. Let $y^m = a^{m-1}x$, which is the equation to a parabolic curve of the m^{th} order: then

$$\frac{dx}{dy} = \frac{my^{m-1}}{a^{m-1}} \text{ and } \frac{d^2x}{dy^2} = \frac{m(m-1)\,y^{m-2}}{a^{m-1}};$$

whence is obtained, $\gamma = \dfrac{(a^{2m-2} + m^2 y^{2m-2})^{\frac{3}{2}}}{m(m-1)\, a^{2m-2} y^{m-2}}$.

If $y = 0$, and $\therefore\ x = 0$, we shall have the radius of curvature at the origin of the co-ordinates *infinite* when m is greater than 2, *finite* when $m = 2$ and *evanescent* when m is less than 2.

181. Thirdly, let s be considered the principal variable, so that x and y may both be treated as functions of it; then by (85), we have, as before,

$$\frac{1}{\gamma} = \frac{d \tan^{-1}\left\{\dfrac{\left(\dfrac{dx}{ds}\right)}{\left(\dfrac{dy}{ds}\right)}\right\}}{ds} = \left\{\frac{\dfrac{d^2x}{ds^2}\dfrac{dy}{ds} - \dfrac{d^2y}{ds^2}\dfrac{dx}{ds}}{\left(\dfrac{dy}{ds}\right)^2}\right\}$$

$$\div \left\{1 + \frac{\left(\dfrac{dx}{ds}\right)^2}{\left(\dfrac{dy}{ds}\right)^2}\right\} = \frac{\dfrac{d^2x}{ds^2}\dfrac{dy}{ds} - \dfrac{d^2y}{ds^2}\dfrac{dx}{ds}}{\left(\dfrac{dx}{ds}\right)^2 + \left(\dfrac{dy}{ds}\right)^2},$$

whence, by virtue of the equation $\left(\dfrac{dx}{ds}\right)^2 + \left(\dfrac{dy}{ds}\right)^2 = 1$ from (127), we immediately obtain $\gamma = \dfrac{1}{\dfrac{d^2x}{ds^2}\dfrac{dy}{ds} - \dfrac{d^2y}{ds^2}\dfrac{dx}{ds}}$,

which may be also written in the following form,

$$\gamma = \frac{ds^3}{d^2x\,dy - d^2y\,dx} = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{d^2x\,dy - d^2y\,dx}.$$

182. Cor. The radius of curvature, determined on this last hypothesis, admits of being exhibited in some very elegant forms.

$$\text{Since } \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1,$$

we have immediately by differentiation, and division by ds,

$$\frac{d^2x}{ds^2}\,\frac{dx}{ds} + \frac{d^2y}{ds^2}\,\frac{dy}{ds} = 0\,.$$

$$\therefore\ \frac{d^2x}{ds^2} = -\frac{d^2y}{ds^2}\,\frac{\left(\frac{dy}{ds}\right)}{\left(\frac{dx}{ds}\right)}, \text{ or } \frac{d^2y}{ds^2} = -\frac{d^2x}{ds^2}\,\frac{\left(\frac{dx}{ds}\right)}{\left(\frac{dy}{ds}\right)}:$$

$$\text{whence we find } \gamma = -\frac{\left(\frac{dx}{ds}\right)}{\left(\frac{d^2y}{ds^2}\right)} \text{ or } \gamma = \frac{\left(\frac{dy}{ds}\right)}{\left(\frac{d^2x}{ds^2}\right)}.$$

$$\text{Also, since } \frac{d^2x}{ds^2}\,\frac{dx}{ds} + \frac{d^2y}{ds^2}\,\frac{dy}{ds} = 0,$$

$$\text{we have likewise } \gamma^2 = \frac{1}{\left\{\frac{d^2x}{ds^2}\,\frac{dy}{ds} - \frac{d^2y}{ds^2}\,\frac{dx}{ds}\right\}^2}$$

$$= \frac{1}{\left\{\frac{d^2x}{ds^2}\,\frac{dy}{ds} - \frac{d^2y}{ds^2}\,\frac{dx}{ds}\right\}^2 + \left\{\frac{d^2x}{ds^2}\,\frac{dx}{ds} + \frac{d^2y}{ds^2}\,\frac{dy}{ds}\right\}^2}$$

$$= \frac{1}{\left\{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right\}\left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2\right\}}$$

$$= \frac{1}{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2};$$

$$\text{and therefore } \gamma = \frac{1}{\sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2}},$$

which may be also written in the following symmetrical form

$$\gamma = \frac{ds^2}{\sqrt{(d^2x)^2 + (d^2y)^2}}.$$

Ex. In a circle whose radius is a and centre the origin of the co-ordinates, we have $x = \sin s$ and $y = \cos s$:

and from each of the expressions just investigated the magnitude of the radius of curvature may be found $= a$.

183. The results obtained in the last three articles may however be easily deduced from that which immediately precedes them, or from (163), by means of the principles of changing the independent variable established in (83) and (84).

Thus, in the formula $\gamma = \frac{(1+p^2)^{\frac{3}{2}}}{q}$, wherein x is considered the principal variable, p and q being the first and second differential coefficients of y, if we put $\frac{1}{p'}$ for p and $-\frac{q'}{p'^3}$ for q, we shall obtain

$$\gamma = \frac{\left\{1 + \frac{1}{p'^2}\right\}^{\frac{3}{2}}}{-\frac{q'}{p'^3}} = -\frac{(1+p'^2)^{\frac{3}{2}}}{q'},$$

in which y is considered the principal variable, p' and q' denoting $\frac{dx}{dy}$ and $\frac{d^2x}{dy^2}$ respectively.

So likewise on the same hypothesis we shall find

$$a = x - \frac{p(1+p^2)}{q} = x + \frac{\frac{1}{p'}\left(1 + \frac{1}{p'^2}\right)}{\frac{q'}{p'^3}} = x + \frac{1+p'^2}{q'},$$

$$\text{and } \beta = y + \frac{1+p^2}{q} = y - \frac{1+\frac{1}{p'^2}}{\frac{q'}{p'^3}} = y - \frac{p'(1+p'^2)}{q'}.$$

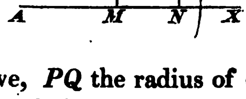
Again, in the same formulæ by substituting $\dfrac{\left(\frac{dy}{ds}\right)}{\left(\frac{dx}{ds}\right)}$ for p,

and therefore $\dfrac{\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}}{\left(\frac{dx}{ds}\right)^2}$ for q, by (84), we shall immediately obtain the remaining results above found.

VII. EVOLUTES AND INVOLUTES.

184. *To find the equation to the curve which is the locus of the centres of the circles of curvature at the different points of a curve.*

Let $AM = x$, $MP = y$, be the co-ordinates of any point P



of the proposed curve, PQ the radius of curvature at P, AN, NQ the co-ordinates of the centre being denoted by α, β as before: then it is manifest that

$$\alpha - x = MN = RQ = PR \tan RPQ = (y - \beta)p:$$

$$\text{but } \alpha - x = PQ \sin RPQ = -\frac{p(1+p^2)}{q}:$$

whence are immediately obtained, the results before found,

$$\alpha = x - \frac{p(1+p^2)}{q} \text{ and } \beta = y + \frac{1+p^2}{q}:$$

wherefore, if by means of the equation of the proposed curve combined with these two, the quantities x and y be eliminated, the relation thence determined between α and β will obviously be the equation of the locus required.

In this proposition it is manifest that the only difficulty we shall have to encounter, is the elimination just mentioned; and the number of cases in which it can be readily effected is very small compared with the number of curves presented to our notice.

The curve whose co-ordinates are α and β is called the *Evolute* of that whose co-ordinates are x and y; and this latter curve is termed the *Involute* of the former, for reasons hereafter explained.

Ex. 1. In the common parabola, we have

$$y^2 = 4ax, \quad p = \frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}} \text{ and } q = -\frac{a^{\frac{1}{2}}}{2x^{\frac{3}{2}}};$$

$$\text{whence } \alpha = 2a + 3x \text{ and } \beta = -\frac{2x^{\frac{3}{2}}}{a^{\frac{1}{2}}};$$

and from these two equations it now remains to eliminate x: from the former $x = \frac{1}{3}(\alpha - 2a)$ and from the latter $x = \dfrac{a^{\frac{1}{3}}\beta^{\frac{2}{3}}}{4^{\frac{1}{3}}}$; whence equating these values we find the equation of the evolute to be

$$\tfrac{1}{4}a\beta^2 = \frac{1}{27}(\alpha - 2a)^3 \text{ or } \beta^2 = \frac{4}{27a}(\alpha - 2a)^3,$$

which is that of a semicubical parabola, the co-ordinates of whose vertex are $2a$ and 0, and whose axis is the axis of x.

Ex. 2. The equation to an ellipse being $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we have $p=-\frac{bx}{a\sqrt{a^2-x^2}}$ and $q=-\frac{ab}{(a^2-x^2)^{\frac{3}{2}}}$; and by means of these expressions we have found in Ex. 2. of (177) that

$$\alpha=\frac{a^2-b^2}{a^4}\,x^3 \text{ and } \beta=-\frac{a^2-b^2}{b^4}\,y^3;$$

wherefore by the help of the three following equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\quad \alpha=\frac{a^2-b^2}{a^4}\,x^3 \quad\text{and}\quad \beta=-\frac{a^2-b^2}{b^4}\,y^3,$$

we have now to eliminate x and y:

$$\text{from the second,}\quad a\alpha=(a^2-b^2)\left(\frac{x}{a}\right)^3,$$

$$\text{and } \therefore\ (a\alpha)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}\left(\frac{x}{a}\right)^2:$$

$$\text{from the third,}\quad b\beta=-(a^2-b^2)\left(\frac{y}{b}\right)^3,$$

$$\text{and } \therefore\ (b\beta)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}\left(\frac{y}{b}\right)^2;$$

whence $(a\alpha)^{\frac{2}{3}}+(b\beta)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}\right\}=(a^2-b^2)^{\frac{2}{3}}$ from the first: that is,

$$(a\alpha)^{\frac{2}{3}}+(b\beta)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}},$$

is the equation of the evolute of an ellipse referred to its principal axes.

If $b=a$, or the ellipse become a circle, we have the equation to the evolute $\alpha^{\frac{2}{3}}+\beta^{\frac{2}{3}}=0$, in which α and β have no corresponding possible values except 0, and the evolute is therefore a point which is the centre of the circle.

In the hyperbola referred to its principal axes, we shall find by a similar process that $(a\alpha)^{\frac{2}{3}}-(b\beta)^{\frac{2}{3}}=(a^2+b^2)^{\frac{2}{3}}$ is the equation of the evolute: and this in the case of the rectangular hyperbola, wherein $b=a$, becomes $\alpha^{\frac{2}{3}}-\beta^{\frac{2}{3}}=(2a)^{\frac{2}{3}}$.

Ex. 3. Let $xy=a^2$; then $p=-\frac{a^2}{x^2}$ and $q=\frac{2a^2}{x^3}$;

$$\text{also, } \alpha=\frac{a^4+3x^4}{2x^3} \text{ and } \beta=\frac{a^4+3y^4}{2y^3},$$

as found in Ex. 3. of (177):

whence in order to eliminate x and y we shall have

$$\beta+\alpha=\frac{a^4+3y^4}{2y^3}+\frac{a^4+3x^4}{2x^3}$$

$$=\frac{1}{2x^3y^3}\{a^4(x^3+y^3)+3x^3y^3(x+y)\}$$

$$=\frac{1}{2a^2}(x+y)^3, \text{ since } xy=a^2;$$

$$\text{and } \beta-\alpha=\frac{1}{2a^2}(x-y)^3, \text{ by a similar process:}$$

$$\therefore\ (\beta+\alpha)^{\frac{1}{3}}=\frac{1}{\sqrt[3]{2a^2}}(x+y) \text{ and } (\beta-\alpha)^{\frac{1}{3}}=\frac{1}{\sqrt[3]{2a^2}}(x-y):$$

$$\text{whence } (\beta+\alpha)^{\frac{1}{3}}+(\beta-\alpha)^{\frac{1}{3}}=\frac{1}{\sqrt[3]{2a^2}}2x,$$

$$\text{and}\quad (\beta+\alpha)^{\frac{1}{3}}-(\beta-\alpha)^{\frac{1}{3}}=\frac{1}{\sqrt[3]{2a^2}}\,2y;$$

$$\therefore\quad (\beta+\alpha)^{\frac{2}{3}}-(\beta-\alpha)^{\frac{2}{3}}=\frac{1}{\sqrt[3]{4a^4}}4xy=\frac{1}{\sqrt[3]{4a^4}}4a^2=(4a)^{\frac{2}{3}},$$

which is the equation of the evolute of a rectangular hyperbola between the asymptotes.

Ex. 4. In the common cycloid $y=a\operatorname{vers}^{-1}\dfrac{x}{a}+\sqrt{2ax-x^2}$,

from which $p=\sqrt{\dfrac{2a-x}{x}}$ and $q=-\dfrac{a}{x\sqrt{2ax-x^2}}$:

and with these by substitution we readily obtain

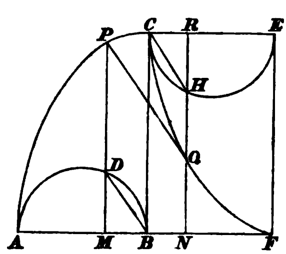
$$\alpha=4a-x\quad\text{and}\quad \beta=a\operatorname{vers}^{-1}\frac{x}{a}-\sqrt{2ax-x^2};$$

whence eliminating x, we find

$$\beta=a\operatorname{vers}^{-1}\left\{\frac{4a-\alpha}{a}\right\}-\sqrt{2a(4a-\alpha)-(4a-\alpha)^2},$$

which is the equation to a cycloid similar and equal to the original one, as may readily be proved.

For, let NQ be produced to R, so that $NR=BC$, and let $CR=x'$, $RQ=y'$, be the co-ordinates of Q from the origin C;



then we shall obviously have

$$x'=BN=AN-AB=\alpha-2a,\quad\text{or}\quad \alpha=2a+x';$$

$$y' = RQ = BC - NQ = \pi a - \beta, \text{ or } \beta = \pi a - y';$$

$\therefore$ the equation above found now becomes

$$\pi a - y' = a \text{ vers}^{-1}\left\{\frac{2a - x'}{a}\right\} - \sqrt{2a(2a - x') - (2a - x')^2}$$

$$= \pi a - a \text{ vers}^{-1}\frac{x'}{a} - \sqrt{2ax' - x'^2};$$

whence $y' = a \text{ vers}^{-1}\dfrac{x'}{a} + \sqrt{2ax' - x'^2}$:

so that the evolute of the cycloid APC is another similar and equal cycloid CQF, situated as in the diagram.

This might have been established upon geometrical principles, by means of the values found in Ex. 4. of (177).

185. *The radius of curvature at any point of a curve is a tangent at the corresponding point of its evolute.*

It has been shewn in the preceding pages, that

$$\alpha = x - \frac{p(1 + p^2)}{q} \text{ and } \beta = y + \frac{1 + p^2}{q};$$

and from these expressions it might by the ordinary process of differentiation be proved that

$$\frac{d\beta}{d\alpha} = -\frac{1}{p},$$

so that

$$\beta - y = -\frac{1}{p}(\alpha - x),$$

the equation to the normal of the proposed curve at the point whose co-ordinates are x, y becomes

$$\beta - y = \frac{d\beta}{d\alpha}(\alpha - x),$$

which is that of the tangent of the evolute at the point whose co-ordinates are α, β; but the same thing may be more conveniently effected as follows.

Retaining the notation hitherto used, we have seen that

$$x-\alpha+(y-\beta)p=0 \text{ and } 1+p^2+(y-\beta)q=0,$$

are equations to the evolute; and this being the locus of the centres of curvature, the quantities α, β must therefore be regarded as variable: but by virtue of the equation $y=f(x)$, these quantities will be also functions of x, and must consequently change their values whenever it does: whence to obtain the ratio between their differentials we have to effect the operation of differentiation on the supposition that y, p, α and β are all functions of x:

from the former of these equations we have

$$1-\frac{d\alpha}{dx}+\frac{dy^2}{dx^2}-\frac{d\beta}{dx}\frac{dy}{dx}+(y-\beta)\frac{d^2y}{dx^2}=0,$$

which, by means of the latter, is reduced to

$$-\frac{d\alpha}{dx}-\frac{d\beta}{dx}\frac{dy}{dx}=0;$$

whence, by changing the independent variable according to the principles explained in (84), we immediately obtain

$$\frac{dx}{dy} \text{ or } \frac{1}{p}=-\frac{d\beta}{d\alpha}:$$

therefore the equation

$$\beta-y=-\frac{dx}{dy}(\alpha-x),$$

which is the equation to the normal of the proposed curve now becomes

$$\beta-y=\frac{d\beta}{d\alpha}(\alpha-x),$$

which is obviously the equation to the tangent of the curve whose co-ordinates are α, β or to the evolute: that is, the radius of curvature of the curve whose co-ordinates are x, y is a tangent to that whose co-ordinates are α, β.

186. *The radius of curvature of a curve varies by the same differences as the corresponding arc of its evolute.*

From the three equations defining the circle of curvature

$$\gamma = \frac{(1+p^2)^{\frac{3}{2}}}{q}, \quad \alpha = x - \frac{p(1+p^2)}{q},$$

$$\text{and } \beta = y + \frac{1+p^2}{q},$$

it would not be difficult to demonstrate directly that

$$d\gamma = \sqrt{d\alpha^2 + d\beta^2};$$

but the same object will be more readily attained by the following process.

The same notation remaining, we have seen that

$$(x-\alpha)^2 + (y-\beta)^2 = \gamma^2,$$

wherein α, β, γ as well as y are manifestly all functions of x, inasmuch as they undergo changes correspondent to any alteration that may take place in it: whence differentiating with respect to x, we find

$$(x-\alpha)\left\{1 - \frac{d\alpha}{dx}\right\} + (y-\beta)\left\{\frac{dy}{dx} - \frac{d\beta}{dx}\right\} = \frac{\gamma d\gamma}{dx},$$

which by virtue of one of the characteristic equations

$$x - \alpha + (y-\beta)\frac{dy}{dx} = 0,$$

is readily reduced to

$$(x-\alpha)\frac{d\alpha}{dx} + (y-\beta)\frac{d\beta}{dx} = -\frac{\gamma d\gamma}{dx}:$$

and by changing the independent variable from x to a by (84), this becomes

$$x - a + (y-\beta)\frac{d\beta}{da} = -\frac{\gamma d\gamma}{da}:$$

$$\therefore\ (x-a)^2 + (y-\beta)^2\left(\frac{d\beta}{da}\right)^2 + 2(x-a)(y-\beta)\frac{d\beta}{da} = \gamma^2\left(\frac{d\gamma}{da}\right)^2:$$

but since by the last article $y-\beta = \frac{d\beta}{da}(x-a)$,

we have $(x-a)(y-\beta)\frac{d\beta}{da} = (x-a)^2\left(\frac{d\beta}{da}\right)^2 = (y-\beta)^2$;

$$\therefore\ \{(x-a)^2 + (y-\beta)^2\}\left\{1 + \left(\frac{d\beta}{da}\right)^2\right\} = \gamma^2\left(\frac{d\gamma}{da}\right)^2$$

$$= \{(x-a)^2 + (y-\beta)^2\}\left(\frac{d\gamma}{da}\right)^2, \text{ or } \left(\frac{d\gamma}{da}\right)^2 = 1 + \left(\frac{d\beta}{da}\right)^2:$$

$$\text{whence } \frac{d\gamma}{da} = \sqrt{1 + \left(\frac{d\beta}{da}\right)^2},$$

which is the differential coefficient of γ with respect to the independent variable a: but by (127), it appears that this latter quantity is also the differential coefficient of the arc of a curve whose co-ordinates are a and β: whence it follows that the differential of the radius of curvature of a curve, is equal to the differential of the arc of its evolute, or that the radius of curvature and the length of the evolute vary by the same differences, and are therefore either equal to one another, or differ by some constant quantity.

Ex. For the Tractrix it has been proved in Ex. 8. of (177), that

$$a - x = \sqrt{a^2 - y^2} \text{ and } \beta - y = \frac{a^2 - y^2}{y};$$

therefore by means of the equation

$$\beta - y = \frac{d\beta}{da}(a - x),$$

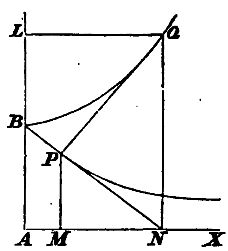
we shall manifestly have

$$\frac{d\beta}{d\alpha} = \frac{\beta - y}{\alpha - x} = \frac{a^2 - y^2}{y\sqrt{a^2 - y^2}} = \frac{\sqrt{a^2 - y^2}}{y};$$

$$\text{whence } \frac{d\gamma}{d\alpha} = \sqrt{1 + \left(\frac{d\beta}{d\alpha}\right)^2} = \frac{a}{y} = \frac{\beta}{a},$$

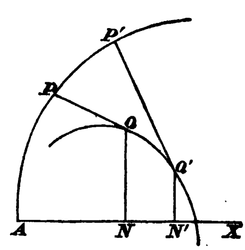
which is a well known differential equation to the catenary whose co-ordinates are α, β and arc γ, a being its parameter.

Thus, if $AN = \alpha$, $NQ = \beta$ and $BQ = QP = \gamma$,



the relation proved to subsist among these quantities demonstrates that the evolute of the tractrix BP is the catenary BQ.

187. The radius of curvature PQ of the curve whose co-ordinates are x, y, being, as proved in (185), a tangent at the



point Q of the curve whose co-ordinates are α, β, has necessarily the same position as an inextensible thread wrapped along the latter curve and disengaged from it as far as Q, at the same time that it is kept constantly stretched. Also, since it has

been proved in (186), that the radius of curvature of the former curve, varies by the same differences as the arc of the latter, if we suppose the points Q and Q' to correspond to P and P' respectively, it is manifest that

$$\text{the arc } QQ' = P'Q' - PQ,$$

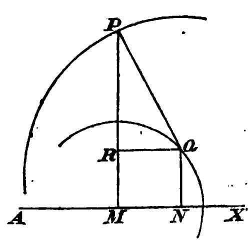
and we may therefore consider the curve whose co-ordinates are x, y to be generated by the unwinding of a thread from that whose co-ordinates are α, β.

This method of tracing the curve AP is somewhat analogous to the description of a circle, the curve AQ performing the office of a centre, and the radius being of variable magnitude: and on this account the curve AQ is called the *Evolute* of the curve AP, which is styled the *Involute*.

188. Cor. 1. If the involute be an algebraic curve, we can always find an expression for its radius of curvature, and therefore for the length of any arc of its evolute, by taking the difference of the radii corresponding to its extremities: and thus we discover that there may exist an infinite number of curves which are *rectifiable*, or of which we can determine exactly the length of any arc.

189. Cor. 2. Conversely, if the equation of the evolute be given, it is easily shewn how that of the corresponding involute may be found.

For, retaining the same figure and notation as before used, we have



$$x = AM = AN - MN = \alpha - QR$$

$$=a - PQ \cos PQR = a - \frac{\gamma\, da}{\sqrt{da^2 + d\beta^2}};$$

$$y = MP = NQ + PR = \beta + PR$$

$$= \beta + PQ \sin PQR = \beta + \frac{\gamma\, d\beta}{\sqrt{da^2 + d\beta^2}}:$$

whence if γ or the length of the evolute be expressed in terms of its co-ordinates, and a, β be then eliminated, the equation resulting between x and y, will be the required equation of the involute.

CHAP. IX.

On the Application of the Differential Calculus to Spirals, or Plane Curves referred to polar Co-ordinates.

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190. If r be the radius vector of a polar curve, and θ be the angle which it makes with a fixed axis, the general equation of the spiral will be of the form

$$r = f(\theta), \text{ or } f(\theta, r) = 0,$$

and the successive differential coefficients of r will be denoted by

$$\frac{dr}{d\theta}, \frac{d^2 r}{d\theta^2}, \frac{d^3 r}{d\theta^3}, \text{ \&c.}:$$

also, if $\theta + h$, $\theta + 2h$, $\theta + 3h$, &c. be succeeding values of the said angle, and r_1, r_2, r_3, &c. represent the corresponding radii vectores, we shall have

$$r_1 = f(\theta + h) = r + \frac{dr}{d\theta}\frac{h}{1} + \frac{d^2 r}{d\theta^2}\frac{h^2}{1.2} + \frac{d^3 r}{d\theta^3}\frac{h^3}{1.2.3} + \text{\&c.};$$

$$r_2 = f(\theta + 2h) = r + \frac{dr}{d\theta}\frac{2h}{1} + \frac{d^2 r}{d\theta^2}\frac{4h^2}{1.2} + \frac{d^3 r}{d\theta^3}\frac{8h^3}{1.2.3} + \text{\&c.};$$

$$r_3 = f(\theta + 3h) = r + \frac{dr}{d\theta}\frac{3h}{1} + \frac{d^2 r}{d\theta^2}\frac{9h^2}{1.2} + \frac{d^3 r}{d\theta^3}\frac{27h^3}{1.2.3} + \text{\&c.}:$$

&c. ....

Similarly, if r_{-1}, r_{-2}, r_{-3}, &c. correspond to $\theta - h$, $\theta - 2h$, $\theta - 3h$, &c. we shall in like manner have

$$r_{-1}=f(\theta-\ h)=r-\frac{dr}{d\theta}\frac{h}{1}+\frac{d^2r}{d\theta^2}\frac{h^2}{1.2}-\frac{d^3r}{d\theta^3}\frac{h^3}{1.2.3}+\&c.;$$

$$r_{-2}=f(\theta-2h)=r-\frac{dr}{d\theta}\frac{2h}{1}+\frac{d^2r}{d\theta^2}\frac{4h^2}{1.2}-\frac{d^3r}{d\theta^3}\frac{8h^3}{1.2.3}+\&c.;$$

$$r_{-3}=f(\theta-3h)=r-\frac{dr}{d\theta}\frac{3h}{1}+\frac{d^2r}{d\theta^2}\frac{9h^2}{1.2}-\frac{d^3r}{d\theta^3}\frac{27h^3}{1.2.3}+\&c.:$$

&c...

and from these we immediately obtain

$$r_1-r\ =\frac{dr}{d\theta}\frac{h}{1}+\frac{d^2r}{d\theta^2}\frac{h^2}{1.2}+\frac{d^3r}{d\theta^3}\frac{h^3}{1.2.3}+\&c.;$$

$$r_2-r_1=\frac{dr}{d\theta}\frac{h}{1}+\frac{d^2r}{d\theta^2}\frac{3h^2}{1.2}+\frac{d^3r}{d\theta^3}\frac{7h^3}{1.2.3}+\&c.;$$

&c...

$$r_{-1}-r\ \ =-\frac{dr}{d\theta}\frac{h}{1}+\frac{d^2r}{d\theta^2}\frac{h^2}{1.2}-\frac{d^3r}{d\theta^3}\frac{h^3}{1.2.3}+\&c.;$$

$$r_{-2}-r_{-1}=-\frac{dr}{d\theta}\frac{h}{1}+\frac{d^2r}{d\theta^2}\frac{3h^2}{1.2}-\frac{d^3r}{d\theta^3}\frac{7h^3}{1.2.3}+\&c.;$$

&c..

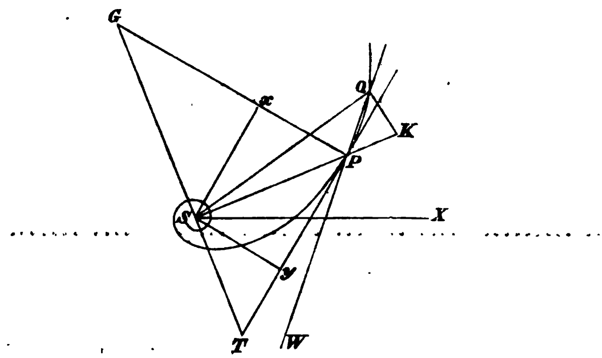
and if h be diminished *sine limite*, it manifestly follows that each of these magnitudes $=\frac{dr}{d\theta}h$.

Wherefore also if s be the arc of the spiral, we shall as in (127) have $\frac{ds}{d\theta}=\sqrt{r^2+\frac{dr^2}{d\theta^2}}$, and $\therefore\ ds=\sqrt{r^2d\theta^2+dr^2}$, as proved in Ex. 2. of (85).

I. TANGENTS.

191. *To find the angle which a straight line cutting a spiral in two points makes with the radius vector.*

Let the straight line WPQ cut the spiral whose Pole is S



in the two points P and Q, $SP=r$, $SQ=r'$, $\angle XSP=\theta$, and $\angle PSQ=h$; also draw QK perpendicular to SP produced, then we have

$$\tan SPW = \tan QPK = \frac{QK}{PK} = \frac{QK}{SK-SP}$$

$$= \frac{SQ \sin h}{SQ \cos h - SP} = \frac{r' \sin h}{r' \cos h - r},$$

$$\text{where } r' = r + \frac{dr}{d\theta}\frac{h}{1} + \frac{d^2 r}{d\theta^2}\frac{h^2}{1.2} + \&c.$$

whence the angle SPW, and consequently the angles SPQ and SQP become known, if the values of θ, r and h be assigned, or the points P and Q of the spiral be given.

192. *To find the angle in which a straight line touching a spiral cuts the radius vector.*

Retaining the figure and notation of the last article we have seen that

$$\tan SPW = \frac{r' \sin h}{r' \cos h - r} = \frac{r' \sin h}{r' - r - 2r' (\sin \tfrac{1}{2} h)^2}$$

$$= \frac{r'}{\left(\dfrac{r'-r}{\sin h}\right) - r' \tan \dfrac{h}{2}};$$

wherefore if the angle PSQ be diminished *sine limite*, so that the points Q and P coincide, and therefore the line QPW come into the position of the tangent PT at P, we shall obtain

$$\tan SPT = \text{limit of } \frac{r'}{\left(\frac{r'-r}{\sin h}\right) - r' \tan\frac{h}{2}} = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{r\,d\theta}{dr}.$$

Also, since the arc and tangent are coincident at the point of contact, this is the angle in which the radius vector cuts the arc of the spiral.

193. Cor. 1. If the angle SPT be called P, since $\tan P = \frac{r\,d\theta}{dr}$, we have $\frac{(\sin P)^2}{1-(\sin P)^2} = \frac{1-(\cos P)^2}{(\cos P)^2} = \frac{r^2 d\theta^2}{dr^2}$;

whence $\sin P = \frac{r\,d\theta}{\sqrt{dr^2 + r^2 d\theta^2}}$ and $\cos P = \frac{dr}{\sqrt{dr^2 + r^2 d\theta^2}}$.

194. Cor. 2. If Sy be drawn perpendicular to the tangent at P, we shall immediately have

$$Sy = SP \sin P = \frac{r^2 d\theta}{\sqrt{dr^2 + r^2 d\theta^2}},$$

$$\text{and } Py = SP \cos P = \frac{r\,dr}{\sqrt{dr^2 + r^2 d\theta^2}}.$$

195. It is often very convenient in practice to know the relation between the radius vector of a spiral, and the perpendicular upon the tangent at the corresponding point; whence calling the perpendicular $Sy = p$, we have in all curves referred to polar co-ordinates

$$p = \frac{r^2}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}}:$$

but this being merely a differential equation, though it may be sufficient to define many of its properties wherein the first differential of the radius vector is concerned, will not serve to determine the several points in the curve.

Hence we have likewise $\frac{d\theta}{dr} = \frac{p}{r\sqrt{r^2-p^2}}$, and by (190), $\frac{ds}{dr} = \frac{r}{\sqrt{r^2-p^2}}$; and if p be put $=r$, the *apsidal* or greatest and least distances may be determined from the equation $\frac{dr}{d\theta}=0$.

Ex. 1. In the Spiral of *Archimedes* $r=a\theta$, $\therefore \frac{dr}{d\theta}=a$:

$$\text{whence } \tan P = \frac{r}{a} \text{ and } p = \frac{r^2}{\sqrt{r^2+a^2}}.$$

Ex. 2. In the Logarithmic Spiral $r=a^\theta$, $\therefore \frac{dr}{d\theta}=ka^\theta=kr$:

$$\text{whence } \tan P = \frac{1}{k} \text{ and } p = \frac{r}{\sqrt{1+k^2}}.$$

Because tan P, and therefore the angle P itself is of a constant magnitude, this curve always cuts the radius vector at the same angle, and is therefore called the *Equiangular* Spiral.

Ex. 3. In the Ellipse about the focus $r=\frac{a(1-e^2)}{1-e\cos\theta}$;

$$\text{whence } \frac{dr}{d\theta} = -\frac{ae(1-e^2)\sin\theta}{(1-e\cos\theta)^2} = -\frac{r}{a}\sqrt{\frac{a^2e^2-(a-r)^2}{1-e^2}};$$

$$\therefore \tan P = -a\sqrt{\frac{1-e^2}{a^2e^2-(a-r)^2}} \text{ and } p = a\sqrt{1-e^2}\sqrt{\frac{r}{2a-r}}.$$

Ex. 4. In a Circle whose radius is a, referred to a pole whose distance from the centre is b, we have

$$r = b\cos\theta + \sqrt{a^2-b^2\sin^2\theta};$$

$$\text{wherefore } \frac{dr}{d\theta} = -\frac{b\sin\theta\,(b\cos\theta+\sqrt{a^2-b^2\sin^2\theta})}{\sqrt{a^2-b^2\sin^2\theta}}$$

$$= -\frac{r\sqrt{4a^2b^2-(a^2+b^2-r^2)^2}}{a^2-b^2+r^2};$$

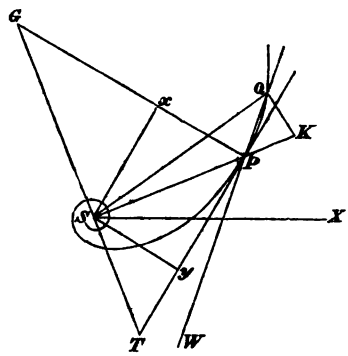
$$\therefore \tan P = -\frac{a^2 - b^2 + r^2}{\sqrt{4a^2b^2 - (a^2 + b^2 - r^2)^2}} \text{ and } p = \frac{a^2 - b^2 + r^2}{2a}.$$

If $b = a$, or the pole be in the circumference, $r = 2a \cos \theta$:

$$\therefore \tan P = -\frac{r}{\sqrt{4a^2 - r^2}} \text{ and } p = \frac{r^2}{2a}.$$

196. *To find the magnitude of the subtangent, and to draw a tangent to a spiral.*

Let S be the pole of the spiral, PT a tangent to it at P, draw ST perpendicular to SP meeting the tangent in



T, then ST is called the *Polar Subtangent:* and the subtangent $ST = SP \tan SPT = r\left(\frac{r\,d\theta}{dr}\right) = \frac{r^2 d\theta}{dr}$.

Hence also the tangent $PT = \sqrt{SP^2 + ST^2}$

$$= \sqrt{r^2 + \frac{r^4 d\theta^2}{dr^2}} = r\sqrt{1 + \frac{r^2 d\theta^2}{dr^2}}.$$

197. Cor. 1. Since $ST = r^2 \frac{d\theta}{dr}$ and $\frac{d\theta}{dr} = \frac{p}{r\sqrt{r^2 - p^2}}$,

we shall manifestly have the subtangent $ST = \frac{pr}{\sqrt{r^2 - p^2}}$;

and therefore if the relation between r and p be given, the tangent may readily be drawn.

198. Cor. 2. If s denote the arc of the spiral, since by (190) $ds = d\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$, we shall have from (194)

$$Py = \frac{\frac{r\,dr}{d\theta}}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}} = \frac{r\,dr}{ds}:$$

whence the relation between r and s being given, if a circle be described upon SP as a diameter, and Py be placed in it equal to the value of $\frac{r\,dr}{ds}$, it will evidently be a tangent to the spiral at the point P.

Ex. 1. In the Reciprocal Spiral $r = a\theta^{-1}$, $\therefore \frac{dr}{d\theta} = -\frac{r^2}{a}$: and the subtangent $ST = -a$, a constant magnitude:

$$\text{also the tangent } PT = r\sqrt{1 + \frac{a^2}{r^2}} = \sqrt{a^2 + r^2}.$$

Ex. 2. In the Logarithmic Spiral $r = a^\theta$, $\therefore \frac{dr}{d\theta} = kr$:

$$\text{whence the subtangent } ST = \frac{r}{k} \propto r \propto SP:$$

$$\text{and the tangent } PT = r\sqrt{1 + \frac{1}{k^2}} = \frac{r}{k}\sqrt{1 + k^2} \propto r \propto SP.$$

Hence it follows that in this spiral the triangle SPT is always similar to itself.

Ex. 3. In a Parabola $p^2 = ar$; wherefore we shall have

$$\text{the subtangent } ST = r\sqrt{\frac{a}{r-a}}:$$

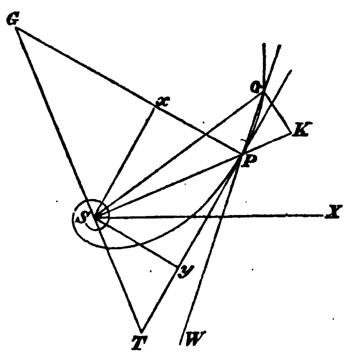
$$\text{and the tangent } PT = r\sqrt{\frac{r}{r-a}}.$$

Ex. 4. Let the arc of a spiral be the measure of the ratio of the radii which intercept it, so that $s = m \log \cdot \left(\frac{r}{a}\right)$; then $ds = \frac{m\,dr}{r}$, whence $Py = \frac{r^2}{m}$, which placed in the circle whose diameter is SP will be a tangent to the spiral.

II. NORMALS.

199. *To find the magnitude of the subnormal, and to draw a normal to a spiral.*

Let PG perpendicular to the spiral at P, and therefore to its tangent at that point, meet the line TSG drawn perpen-



dicular to SP in G, then SG is called the *Polar Subnormal*: and the subnormal SG

$$= SP \tan SPG = SP \cot SPT = r\,\frac{dr}{r\,d\theta} = \frac{dr}{d\theta}:$$

also the normal $PG = \sqrt{SP^2 + SG^2} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \frac{ds}{d\theta}$:

Hence, also, if Sx be drawn from the pole perpendicular to the normal, we shall have

$$Sx = SP \sin SPG = SP \cos P = \frac{r\,dr}{\sqrt{dr^2 + r^2 d\theta^2}} = Py,$$

$$Px - SP \cos SPG = SP \sin P = \frac{r^2 d\theta}{\sqrt{dr^2 + r^2 d\theta^2}} = Sy.$$

Ex. 1. In the Spiral of *Archimedes* $r = a\theta$; whence the subnormal $SG = a$, a constant quantity, and the normal

$$PG = \sqrt{a^2 + r^2}.$$

Ex. 2. Let $\theta = \log\left(\frac{a + \sqrt{a^2 - r^2}}{r}\right)$, $\therefore \frac{d\theta}{dr} = -\frac{a}{r\sqrt{a^2 - r^2}}$:

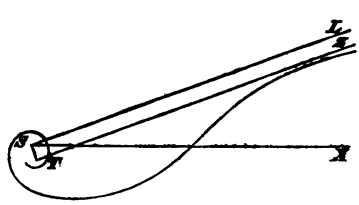
whence the subnormal $SG = -\frac{r\sqrt{a^2 - r^2}}{a}$:

and the normal $PG = \frac{r\sqrt{2a^2 - r^2}}{a}$.

III. ASYMPTOTES.

200. *To find whether a spiral admits of an asymptote, and to determine its position.*

Let the equation to the proposed spiral be $r = f(\theta)$; then $\theta = f^{-1}(r)$, and the magnitude of the subtangent $= \frac{r^2 d\theta}{dr}$: wherefore if these quantities do not become infinite when the radius vector r is indefinitely increased, the spiral admits of an



asymptote, the position of which may therefore be determined;

for, if SL be the direction of the radius vector when infinite, and the subtangent ST be drawn perpendicular to it, TZ passing through T parallel to SL will manifestly be the asymptote.

Ex. 1. In the Reciprocal Spiral $r = a\theta^{-1}$, and $\therefore\ \theta = \frac{a}{r}$, which becomes $= 0$, when $r = \infty$;

also, the subtangent $= \frac{r^2 d\theta}{dr} = -a$, a constant magnitude, whatever be the value of r, and consequently when r is infinite: wherefore, if SX be the line from which θ is measured, it is manifestly the direction of the radius vector when infinite, and ST being made $= -a$, the line TZ drawn through the point T parallel to SX will be the asymptote required.

Ex. 2. In the Lituus $r = a\theta^{-\frac{1}{2}}$, or $\theta = \frac{a^2}{r^2}$; wherefore if r be made $= \infty$, we shall manifestly have $\theta = 0$:

also, the subtangent $= -\frac{2a^2}{r}$, which likewise $= 0$, if $r = \infty$, and therefore the line SX, from which the angle θ is measured, is the asymptote of the spiral.

Ex. 3. In the hyperbola referred to the focus we have

$$r = \frac{a(e^2 - 1)}{1 - e\cos\theta};$$

and from this, if $r = \infty$, we find $\cos\theta = \frac{1}{e}$; also, the subtangent $= -\frac{a(e^2 - 1)}{e\sin\theta}$, which when $r = \infty$, gives $ST = -a\sqrt{e^2 - 1}$; hence therefore if the angle LSX be made such that $\cos LSX = \frac{1}{e}$, and ST be drawn perpendicular to SL, and made equal to $-a\sqrt{e^2 - 1}$, the line TZ parallel to SL will be the asymptote required.

Let ZT produced backwards meet the axis in C, then since

$$SC = \frac{ST}{\sin LSX} = -\frac{a\sqrt{e^2-1}}{\sqrt{1-\frac{1}{e^2}}} = -\frac{ae\sqrt{e^2-1}}{\sqrt{e^2-1}} = -ae,$$

it is evident that the asymptote passes through the centre:

$$\text{and } \tan ZCX = \frac{\sin ZCX}{\cos ZCX} = \frac{\sqrt{1-\frac{1}{e^2}}}{\frac{1}{e}} = \sqrt{e^2-1}$$

$$= \frac{\sqrt{a^2e^2-a^2}}{a} = \frac{\sqrt{SC^2-AC^2}}{AC} = \frac{BC}{AC} = \frac{b}{a},$$

as may readily be deduced from what has been previously proved in the first example of article (153).

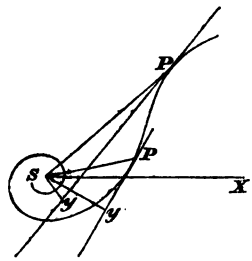
Ex. 4. In the Cissoid of *Diocles* $r = \dfrac{2a(\sin\theta)^2}{\cos\theta}$, and therefore if $r = \infty$, we shall have $\cos\theta = 0$, or $\theta = \dfrac{\pi}{2}$:

also, the value of the subtangent $= \dfrac{2a(\sin\theta)^2}{2-(\sin\theta)^2}$, which, when $r = \infty$, or $\theta = \dfrac{\pi}{2}$, gives $ST = 2a$: hence, if SL be drawn perpendicular to SX and ST be taken $= 2a$, TZ drawn parallel to SL will be the asymptote: and it may be observed, that this entirely agrees with what has been proved in the second example of article (151) in the preceding Chapter.

IV. DIRECTION OF CURVATURE.

201. *To determine the position of a spiral with respect to its pole or radius vector.*

When a spiral is concave towards its pole or radius vector, it is obvious from an examination of the figure, that the per-



pendicular upon the tangent $Sy = p$ increases or decreases according as the radius vector $SP = r$ is increased or diminished: hence, therefore if $p = f(r)$ be the equation to the spiral, and the increment of r be called k, we shall have

$$\frac{\Delta Sy}{\Delta SP} = \frac{dp}{dr} + \frac{d^2p}{dr^2}\frac{k}{1.2} + \frac{d^3p}{dr^3}\frac{k^2}{1.2.3} + \&c.,$$

the limits of which being taken, it is evident that when k is diminished *sine limite*, the differential coefficient $\frac{dp}{dr}$ must be positive.

In the same manner when the spiral is convex towards its pole or radius vector, it appears that Sy increases or decreases whilst SP decreases or increases, and therefore that $\frac{dp}{dr}$ must be negative.

Hence conversely, a spiral will be concave or convex towards its pole or radius vector, according as $\frac{dp}{dr}$ is positive or negative.

Ex. 1. In the Spiral of *Archimedes* where $r = a\theta$,

we have $p = \frac{r^2}{\sqrt{r^2 + a^2}}$; whence $\frac{dp}{dr} = \frac{r(2a^2 + r^2)}{(a^2 + r^2)^{\frac{3}{2}}}$,

which is positive whatever be the value of r, and therefore this spiral is always concave towards its radius vector.

Ex. 2. In an Hyperbola, the pole being situated in the centre, we have

$$r = a\sqrt{\frac{e^2-1}{e^2\cos^2\theta - 1}};$$

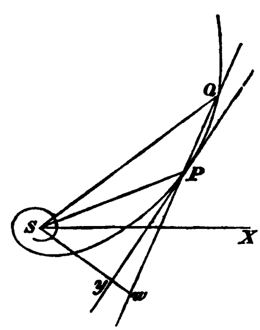
wherefore $p = \dfrac{ab}{\sqrt{r^2-a^2+b^2}}$, and $\dfrac{dp}{dr} = -\dfrac{abr}{(r^2-a^2+b^2)^{\frac{3}{2}}}$;

and the latter quantity being always negative, proves that the arc of an hyperbola is always convex towards its centre, and therefore towards the radius vector from the centre.

V. CONTACT AND OSCULATION.

202. Def. If the perpendicular p let fall upon the tangent from the pole of the spiral be considered as a function of the radius vector r whose increment is k, then will the increment of p, or Δp be represented by

$$\frac{dp}{dr}\frac{k}{1} + \frac{d^2p}{dr^2}\frac{k^2}{1.2} + \frac{d^3p}{dr^3}\frac{k^3}{1.2.3} + \text{\&c.};$$



and if QP and Sy be produced till they meet in w, we shall have

$$\tan yPw = \frac{yw}{Py} = \frac{Sw - Sy}{Py}:$$

now, taking the limits of both sides of this equation, observing that an arc and its tangent are ultimately equal, and that $Sw - Sy$ is ultimately $= \Delta p$, we manifestly obtain

$$\angle yPw = \left\{\frac{dp}{dr}\,\frac{k}{1} + \frac{d^2p}{dr^2}\,\frac{k^2}{1\,.\,2} + \frac{d^3p}{dr^3}\,\frac{k^3}{1\,.\,2\,.\,3} + \&c.\right\}\frac{1}{\sqrt{r^2-p^2}},$$

whose ultimate value is the inclination of two tangents to each other at the distances r and $r+k$.

Whence, if there be two polar curves whose equations are $p=f(r)$ and $p'=F(r')$, and ϕ, ϕ' denote the corresponding values of the angle above-mentioned in the two curves for the common increment k, we shall have

$$\phi = \left\{\frac{dp}{dr}\,\frac{k}{1} + \frac{d^2p}{dr^2}\,\frac{k^2}{1\,.\,2} + \frac{d^3p}{dr^3}\,\frac{k^3}{1\,.\,2\,.\,3} + \&c.\right\}\frac{1}{\sqrt{r^2-p^2}},$$

$$\phi' = \left\{\frac{dp'}{dr'}\,\frac{k}{1} + \frac{d^2p'}{dr'^2}\,\frac{k^2}{1\,.\,2} + \frac{d^3p'}{dr'^3}\,\frac{k^3}{1\,.\,2\,.\,3} + \&c.\right\}\frac{1}{\sqrt{r'^2-p'^2}}:$$

wherefore when the spirals have a common pole, and the radii vectores make equal angles with the same fixed axis:

if $r'=r$, the spirals have a point of intersection:

if $r'=r$ and $p'=p$, the spirals have a common tangent, and a contact of the *first* order:

if $r'=r$, $p'=p$ and $\dfrac{dp'}{dr'} = \dfrac{dp}{dr}$, the spirals have a common tangent and a contact of the *second* order: and so on.

And it may be proved precisely as in article (160), that the angle contained between the spirals in each of these cases is infinitely greater than in any of the succeeding: and hence also as in curves referred to rectangular co-ordinates, it may be made to appear that no spiral can pass between two others which have a contact of an order superior to that which it has with either of them.

203. *To find the conditions necessary that a circle may have contact of the second order with a proposed spiral.*

Let a be the distance of the centre of the circle from the pole of the spiral, γ the radius; then by the fourth example of article (195), we have

$$p' = \frac{\gamma^2 - a^2 + r'^2}{2\gamma}, \text{ and } \therefore \frac{dp'}{dr'} = \frac{2r'}{2\gamma} = \frac{r'}{\gamma};$$

$$\text{but since } r' = r,\ p' = p \text{ and } \frac{dp'}{dr'} = \frac{dp}{dr},$$

$$\text{it follows that } \frac{dp}{dr} = \frac{r}{\gamma}, \text{ and therefore that } \gamma = \frac{r\,dr}{dp},$$

which is the value of the radius of the circle:

$$\text{also, } a^2 = r'^2 + \gamma^2 - 2p'\gamma = r^2 + \frac{r^2 dr^2}{dp^2} - \frac{2prdr}{dp},$$

$$\text{and } \therefore a = \sqrt{r^2 - 2pr\frac{dr}{dp} + r^2\frac{dr^2}{dp^2}},$$

which is the distance of the centre of the circle from the pole of the spiral; and thus the magnitude and position of the circle are expressed in terms of the values of p and r belonging to any proposed point of the spiral.

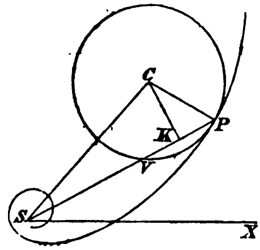
204. The same kind of reasoning, as has been employed in (163), demonstrates that no other circle can in general have with the spiral a contact of a higher order than the one thus determined, which is therefore the *Osculating Circle.*

Moreover, by assuming the differential coefficients of superior orders to be equal in two spirals, *osculating* spirals of given species and succeeding orders may be determined by means of the principles laid down in (166): and it might easily be proved, as in (170), that when the contact is of an odd order, there is only contact, and when it is of an even order, there is both contact and intersection.

VI. CIRCLE OF CURVATURE.

205. *To find expressions for determining the magnitude and position of the circle of curvature to any point of a spiral, and also the chord passing through the pole.*

The circle of curvature being, as in (173), the osculating circle at the same point, if a be the distance SC of the centre



of the circle from the pole of the spiral, and γ the radius CP, we have seen that

$$\gamma = \frac{r\,dr}{dp}, \text{ and } a = \sqrt{r^2 - 2pr\frac{dr}{dp} + \frac{r^2 dr^2}{dp^2}},$$

which, with any assigned point P, determine the position of its centre and the magnitude of its radius:

also, if CK be drawn perpendicular to SP, it is obvious that the *Chord of Curvature* passing through the pole of the spiral

$$= 2PK = 2CP\frac{Sy}{SP}$$

$$= \frac{2r\,dr}{dp}\,\frac{p}{r} = \frac{2p\,dr}{dp}.$$

Ex. 1. In the Logarithmic Spiral $p = \dfrac{r}{\sqrt{1+k^2}}$ by (195),

whence the radius of curvature $CP = \dfrac{rdr}{dp} = r\sqrt{1+k^2}$;

and the chord of curvature $PV = \dfrac{2pdr}{dp} = 2r$.

Ex. 2. In an Ellipse whose polar co-ordinates are referred to the focus, we have by (195)

$$p = a\sqrt{1-e^2}\sqrt{\frac{r}{2a-r}};$$

$$\text{wherefore } CP = \frac{(2ar-r^2)^{\frac{3}{2}}}{a^2\sqrt{1-e^2}} = a(1-e^2)\frac{r^3}{p^3};$$

and $PV = \dfrac{2r(2a-r)}{a} = \dfrac{2b'^2}{a}$, if b' be the semi-conjugate diameter to the point P.

Ex. 3. In an Hyperbola about the centre, we have

$$p = \frac{ab}{\sqrt{r^2-a^2+b^2}};$$

$$\text{whence } CP = -\frac{(r^2-a^2+b^2)^{\frac{3}{2}}}{ab} = -\frac{a^2b^2}{p^3},$$

the negative sign shewing that the radius of curvature and the radius vector lie on different sides of the curve:

$$\text{also, } PV = -\frac{2(r^2-a^2+b^2)}{r} = -\frac{2b'^2}{r},$$

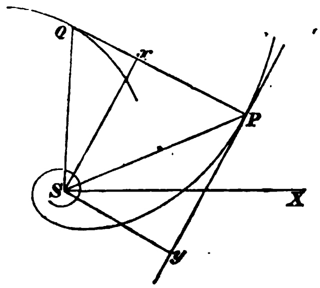
if b' be the semi-conjugate diameter to the point P.

VII. EVOLUTES AND INVOLUTES.

206. *To find the equation to the Evolute of a polar curve.*

Let $SP=r$, $Sy=p$ and suppose $p=f(r)$ to be the equation of the spiral: let PQ be the radius of curvature at P,

which by article (185) is a tangent to the evolute; draw Sx



perpendicular to it and put $SQ = r'$, $Sx = p'$: then we have

$$p' = Sx = Py = \sqrt{SP^2 - Sy^2} = \sqrt{r^2 - p^2},$$

$$\text{and } r' = SQ = \sqrt{Sx^2 + Qx^2} = \sqrt{r^2 - p^2 + \left(\frac{r\,dr}{dp} - p\right)^2}$$

$$= \sqrt{r^2 - 2pr\frac{dr}{dp} + r^2\frac{dr^2}{dp^2}};$$

wherefore if by means of these two equations and that of the proposed spiral $p = f(r)$, the quantities p and r be eliminated, the equation between p' and r' will be obtained.

It is manifest from article (186) that PQ is either equal to the length of the evolute, or differs from it by a constant magnitude.

Ex. 1. In the Logarithmic Spiral $p = \dfrac{r}{\sqrt{1+k^2}}$;

$$\text{wherefore } p' = \sqrt{r^2 - p^2} = \sqrt{r^2 - \frac{r^2}{1+k^2}} = \frac{kr}{\sqrt{1+k^2}},$$

and $r' = \sqrt{r^2 - 2r^2 + r^2(1+k^2)} = kr$; whence

$$\frac{p'}{r'} = \frac{kr}{\sqrt{1+k^2}}\,\frac{1}{kr} = \frac{1}{\sqrt{1+k^2}}, \text{ and } p' = \frac{r'}{\sqrt{1+k^2}};$$

that is, the evolute of a logarithmic spiral is a logarithmic spiral similar and equal to it.

Ex. 2. Let the equation to the spiral be $p = \sqrt{r^2 - a^2}$;

$$\text{then we have } p' = \sqrt{r^2 - p^2} = \sqrt{r^2 - r^2 + a^2} = a,$$

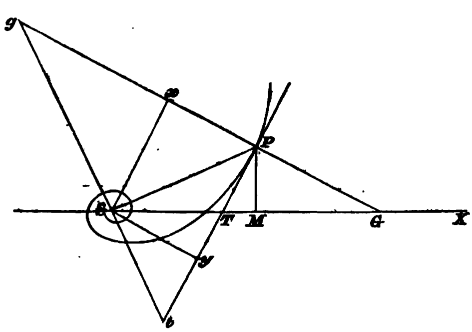
$$\text{and } r' = \sqrt{r^2 - 2(r^2 - a^2) + r^2 - a^2} = a;$$

and from this it appears that $p' = r'$, which is the equation to a circle.

The spiral whose equation is $p = \sqrt{r^2 - a^2}$, is therefore called the *Involute of the Circle*, whose radius is a.

207. Cor. The method of determining the Involute from the Evolute is analogous to that explained in (189).

208. Without entering upon the investigations comprised in this chapter, the principles explained and the formulæ deduced in the last, together with the transformation of rectangular into polar co-ordinates, would have been amply sufficient for the determination of all the properties of spirals above given. Thus, retaining the notation previously used, and making S the origin of rectangular co-ordinates, we have



$$x = SM = SP \cos PSM = r \cos \theta,$$

$$y = PM = SP \sin PSM = r \sin \theta:$$

whence $\tan SPT = \tan (SPM - TPM)$

$$= \frac{\tan SPM - \tan TPM}{1 + \tan SPM \tan TPM} = \frac{\frac{x}{y} - \frac{dx}{dy}}{1 + \frac{xdx}{ydy}}$$

$$= \frac{xdy - ydx}{ydy + xdx} = \frac{x\frac{dy}{d\theta} - y\frac{dx}{d\theta}}{y\frac{dy}{d\theta} + x\frac{dx}{d\theta}}:$$

but $\frac{dx}{d\theta} = \cos\theta \frac{dr}{d\theta} - r\sin\theta$ and $\frac{dy}{d\theta} = \sin\theta\frac{dr}{d\theta} + r\cos\theta$:

and these values being substituted in the expression above found, give

$$\tan SPT = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{rd\theta}{dr},$$

as obtained in (192).

Hence also, $Sy = SP \sin SPy = \frac{r^2 d\theta}{\sqrt{r^2 d\theta^2 + dr^2}}$,

and $Sx = SP \cos SPy = \frac{rdr}{\sqrt{r^2 d\theta^2 + dr^2}}$:

and if gSt be drawn perpendicular to the radius vector SP, meeting the tangent and normal, or those lines produced, in t and g respectively, we shall have

the polar subtangent $St = SP \tan SPt = \frac{r^2 d\theta}{dr}$,

and the polar subnormal $Sg = SP \cot SPt = \frac{dr}{d\theta}$:

as before proved.

209. The lines St and Sg here designated the polar subtangent and subnormal, are essentially different from the

lines MT and MG the subtangent and subnormal of the curve referred to rectangular ordinates: and indeed except for the simplicity and practical convenience of the formulæ by which they are represented, it would scarcely have been necessary to introduce the consideration of these additional lines at all; for if the position of the fixed axis of the spiral be known or assumed, the rectangular subtangent and subnormal may with great facility be expressed in terms of the polar co-ordinates, and thence may the tangent and normal readily be drawn. Thus,

$$\text{the subtangent } MT = y\frac{dx}{dy} = y\frac{\left(\frac{dx}{d\theta}\right)}{\left(\frac{dy}{d\theta}\right)}$$

$$= r\sin\theta\,\frac{\cos\theta\frac{dr}{d\theta} - r\sin\theta}{\sin\theta\frac{dr}{d\theta} + r\cos\theta}$$

$$= \frac{r\sin\theta\cos\theta\,dr - r^2\sin^2\theta\,d\theta}{\sin\theta\,dr + r\cos\theta\,d\theta}:$$

$$\text{and the subnormal } MG = y\frac{dy}{dx} = y\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$$

$$= r\sin\theta\,\frac{\sin\theta\frac{dr}{d\theta} + r\cos\theta}{\cos\theta\frac{dr}{d\theta} - r\sin\theta}$$

$$= \frac{r\sin^2\theta\,dr + r^2\sin\theta\cos\theta\,d\theta}{\cos\theta\,dr - r\sin\theta\,d\theta}.$$

Ex. The polar Equation to the common parabola being

$$r = \frac{a}{(\sin\frac{1}{2}\theta)^2},$$

we have

$$\frac{dr}{d\theta} = -\frac{a\cos\frac{1}{2}\theta}{(\sin\frac{1}{2}\theta)^3}:$$

whence, by substitution in the formulæ just deduced and by reduction, we obtain $MT = -2a(\cot\frac{1}{2}\theta)^2$, and $MG = -2a$.

210. The length of the perpendicular let fall upon the tangent from the origin of the co-ordinates, may likewise be expressed in terms of the polar co-ordinates by a similar process.

For, by (137), $p = AK = \dfrac{y\,dx - x\,dy}{\sqrt{dx^2 + dy^2}}$

$$= \frac{r\sin\theta\cos\theta\,dr - r^2\sin^2\theta\,d\theta - r\sin\theta\cos\theta\,dr - r^2\cos^2\theta\,d\theta}{\sqrt{\cos^2\theta\,dr^2 + r^2\sin^2\theta\,d\theta^2 + r^2\cos^2\theta\,d\theta^2 + \sin^2\theta\,dr^2}}$$

$$= -\frac{r^2\,d\theta}{\sqrt{r^2\,d\theta^2 + dr^2}} = -\frac{r^2}{\sqrt{r^2 + \dfrac{dr^2}{d\theta^2}}},$$

as proved in (194), the algebraical sign, which affects not the magnitude of the proposed line, being disregarded.

211. Cor. 1. Hence we have immediately by differentiation,

$$\frac{dp}{dr} = \frac{r^3 - r^2\dfrac{d^2r}{d\theta^2} + 2r\dfrac{dr^2}{d\theta^2}}{\left(r^2 + \dfrac{dr^2}{d\theta^2}\right)^{\frac{3}{2}}}:$$

but it has been proved in (201) that the spiral is concave or convex towards its pole according as $\dfrac{dp}{dr}$ is positive or negative:

whence it obviously follows that this will also be the case according as

$$\frac{r^3 - r^2\dfrac{d^2 r}{d\theta^2} + 2r\dfrac{dr^2}{d\theta^2}}{\left(r^2 + \dfrac{dr^2}{d\theta^2}\right)^{\frac{3}{2}}},$$

is positive or negative: that is, according as $r^3 - r^2\dfrac{d^2r}{d\theta^2} + 2r\dfrac{dr^2}{d\theta^2}$ is positive or negative, or according

as $r^3 + 2r\dfrac{dr^2}{d\theta^2}$ is greater or less than $r^2\dfrac{d^2r}{d\theta^2}$,

as $r + \dfrac{2dr^2}{rd\theta^2}$ is greater or less than $\dfrac{d^2r}{d\theta^2}$,

as $\dfrac{d^2r}{d\theta^2}$ is less or greater than $r + \dfrac{2dr^2}{rd\theta^2}$.

212. Cor. 2. From the expression just found for $\dfrac{dp}{dr}$, we have immediately by means of (203),

$$\gamma = \frac{rdr}{dp} = \frac{\left(r^2 + \dfrac{dr^2}{d\theta^2}\right)^{\frac{3}{2}}}{r^2 - r\dfrac{d^2r}{d\theta^2} + 2\dfrac{dr^2}{d\theta^2}},$$

which might also have been derived by substitution from the expression $\gamma = \dfrac{(dx^2 + dy^2)^{\frac{3}{2}}}{dyd^2x - dxd^2y}$.

Similarly, the co-ordinates of the centre of curvature, as well as the chord passing through any given point, may be expressed in terms of the polar co-ordinates.

Ex. In the common parabola $r = a\sec^2\frac{1}{2}\theta = a(1 + t^2)$, if $t = \tan\frac{1}{2}\theta$:

$$\therefore \frac{dr}{d\theta} = 2at\frac{dt}{d\theta} = at(1+t^2) = rt:$$

whence we have $r + \dfrac{2dr^2}{rd\theta^2} = a(1+t^2)(1+2t^2)$:

$$\text{again, } \frac{d^2 r}{d\theta^2} = t\frac{dr}{d\theta} + r\frac{dt}{d\theta}$$

$$= at^2(1+t^2) + \tfrac{1}{2}a(1+t^2)(1+t^2)$$

$$= \tfrac{1}{2}a(1+t^2)(1+3t^2):$$

now it is evident that $\frac{1}{2}a(1+t^2)(1+3t^2)$ is always less than $\frac{1}{2}a(1+t^2)(2+4t^2)$ or $a(1+t^2)(1+2t^2)$: therefore, $\dfrac{d^2 r}{d\theta^2}$ is always less than $r + \dfrac{2dr^2}{rd\theta^2}$, and consequently the arc of the parabola has its concavity always turned towards the focus considered as its pole.

$$\text{Also, } \gamma = \frac{\{a^2(1+t^2)^2 + a^2t^2(1+t^2)^2\}^{\frac{3}{2}}}{a^2(1+t^2)^2 - \frac{1}{2}a^2(1+3t^2)(1+t^2)^2 + 2a^2t^2(1+t^2)^2}$$

$$= 2a(1+t^2)^{\frac{3}{2}} = 2a\sec^3\tfrac{1}{2}\theta = \frac{2r^{\frac{3}{2}}}{a^{\frac{1}{2}}}, \text{ as it ought.}$$

CHAP. X.

On the Analytical Characters of the Singular Points of Plane Curves, and the methods of determining their Natures and Positions.

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213. DEF. IN curves whose natures are expressed generally by the equation $y = f(x)$ or $f(x, y) = 0$, it is evident that by assigning different values to the principal variable, different values will in general be assigned to the dependent variable as well as to its differential coefficients: and when any of these functions attains a value attended with some peculiarity either in its value or its form, the corresponding point in the curve will be distinguished by something peculiar in its character. Points of this description are usually denominated *Singular Points*, the principal of which are characterized by the circumstances explained and exemplified in the following articles, and for the sake of simplicity, they shall here be distinguished into *Simple* and *Multiple* points, according as they belong to one or more branches of the curve.

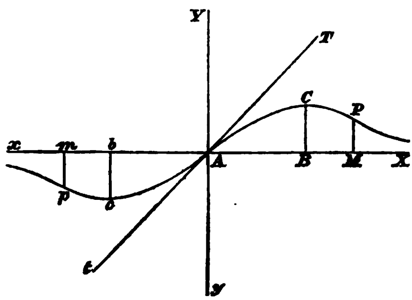
SIMPLE POINTS.

(1) *Points corresponding to evanescent Ordinates.*

214. If when any value is assigned to the abscissa x, the corresponding value of the ordinate y become $= 0$, it is obvious that the curve meets the axis of x, and it is manifest from (133) that the angle at which the concourse takes place may be found from the equation $\tan X = \frac{dy}{dx}$, by substituting for x the said assigned magnitude.

Hence also the positions of the evanescent ordinates may be ascertained by the solution of the equation $y=f(x)=0$; and it is obvious that a similar process will enable us to determine the same circumstances with respect to the axis of y.

Ex. 1. Let the curve proposed be defined by the equation $y=\frac{x}{1+x^2}$: then the equation $y=0$ gives immediately $x=0$ and $x=\pm\infty$, so that the curve meets the axis of x in the origin and also at an infinite distance on each side of it:



also, since $\tan X=\frac{dy}{dx}=\frac{1-x^2}{(1+x^2)^2}=\cot Y$, if we suppose $x=0$, we shall have $\tan TAX=1=\cot TAY$, and therefore,

$$\angle TAX=45^0=\angle TAY;$$

or in other words, this curve intersects each of the co-ordinate axes in the origin at an angle of 45^0: but if $x=\pm\infty$, we shall have $\tan X=0=\cot Y$, so that at the said infinite distances the axis of x touches the curve.

Ex. 2. In an Ellipse whereof one of the foci is the origin of co-ordinates, and the axis-major coincides with the axis of x, we have

$$y=\sqrt{1-e^2}\sqrt{a^2(1-e^2)+2aex-x^2},$$

a and e being the semi-axis-major and eccentricity respectively: now if $y=0$, or the curve meet the axis of x, we must make

$a^2(1-e^2)+2aex-x^2=0$, from which are readily found $x=a(1+e)$ and $x=-a(1-e)$: again, if $x=0$, we have $y=\pm a(1-e^2)$: and these determine the points of intersection of the curve with the axes of x and y respectively:

but from the same equation, $\dfrac{dy}{dx}=\dfrac{\sqrt{1-e^2}\,(ae-x)}{\sqrt{a^2(1-e^2)+2aex-x^2}}$;

therefore, if $x=a(1+e)$ we find $\tan X=\infty=\tan 90^0$, and if $x=-a(1-e)$, we have $\tan X=\infty=\tan 90^0$; that is, this ellipse intersects the axis of x in the extremities of the axis-major at right-angles:

also, if $x=0$ and $y=a(1-e^2)$, we find $\cot Y=e$;

if $x=0$ and $y=-a(1-e^2)$, we have $\cot Y=-e$:

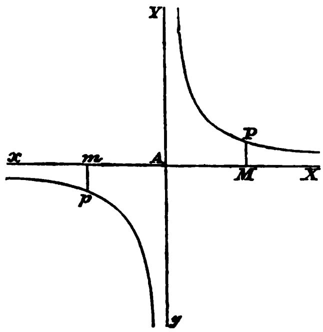
or this curve cuts the axis of y in angles whose cotangents are e and $-e$ respectively: in other words, the ellipse at the extremities of its *latus rectum* is inclined to the axis of y at these angles.

(2) *Points corresponding to infinite Ordinates.*

215. Whenever a finite magnitude assigned to x renders the value of y indefinitely great, the ordinate corresponding thereto is infinite and becomes an asymptote to the curve, because as y can become infinite only by the evanescence of the denominator of the quantity which expresses it, the value of the trigonometrical tangent $\dfrac{dy}{dx}$ will be infinite likewise: or the tangent at an infinite distance, which is then an asymptote to the curve, will be perpendicular to the axis of x.

Hence also, the positions of the infinite ordinates may be ascertained by equating to zero the denominator of the expression for y, and then finding the roots of the equation. Similarly for the axis of y.

Ex. 1. Let the equation of the curve be $xy = a^2$: then by assuming $x = 0$, we find $y = \infty$, and by making $y = 0$, we get $x = \infty$:



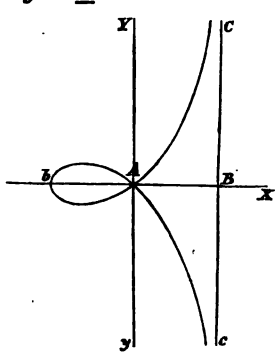
also $\dfrac{dy}{dx} = -\dfrac{a^2}{x^2} = -\infty = -\tan YAX = -\tan 90^0$, if $x = 0$,

and $\dfrac{dx}{dy} = -\dfrac{a^2}{y^2} = -\infty = -\tan XAY = -\tan 90^0$, if $y = 0$;

whence it appears that the co-ordinate axes of y and x are asymptotes to the curve in their respective directions.

Ex. 2. Let $(a - x)y^2 = ax^2 + x^3$ be the equation of the curve proposed: then will $y = \pm x\sqrt{\dfrac{a + x}{a - x}}$:

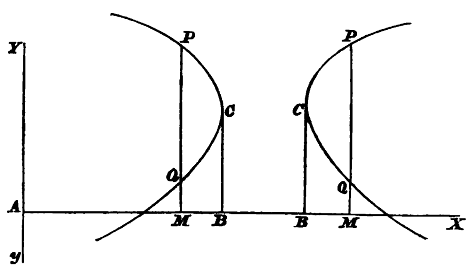
whence if $a - x$ be assumed $= 0$ or $x = a$, we shall have corresponding thereto $y = \pm\infty$:



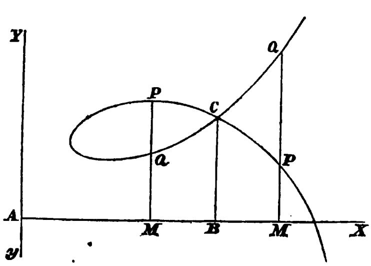
also, $\dfrac{dy}{dx} = \pm \dfrac{a^2 + ax - x^2}{a^2 - x^2} \sqrt{\dfrac{a+x}{a-x}}$ will likewise become indefinitely great on the same hypothesis: that is, if A be the origin of co-ordinates and AB be made $= a$, the corresponding double ordinate CBc drawn through the point B, and produced indefinitely both ways, becomes an asymptote to the curve, the angles CBX, cBX being both right angles, because their trigonometrical tangents are infinite.

(3) *Points corresponding to equal Ordinates.*

216. If when a given value is assigned to the abscissa x, it appears that the corresponding ordinate y has two or more equal values, this ordinate will obviously limit the curve in the direction of the axis of x if either its succeeding or preceding value become imaginary, as in the case of the ordinates BC in the following diagram:



or will pass through a point attended with some such peculiarity as characterizes the point C in the diagram underneath,

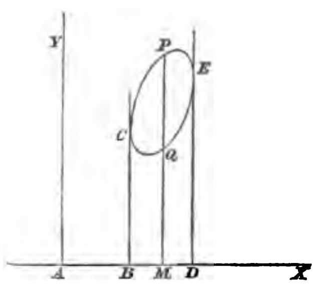


should its succeeding and preceding values be both real quantities.

Points of this description may manifestly be determined by equating to each other the different values of y: and similar remarks will be equally applicable if the abscissæ be measured along the axis of y.

Ex. 1. Let the curve proposed be an ellipse defined by the equation $y^2-2xy+3x^2-2y-4x+5=0$; from which we deduce immediately

$$y=x+1\pm\sqrt{-2x^2+6x-4}=x+1\pm\sqrt{2(x-1)(2-x)}:$$



therefore if x be made $=1$, we shall have each of the two values of $y=1+1=2$: also, if $x=2$, each of the two values of $y=2+1=3$: and for the values of x less than 1 and greater than 2, the corresponding values of the ordinate become impossible:

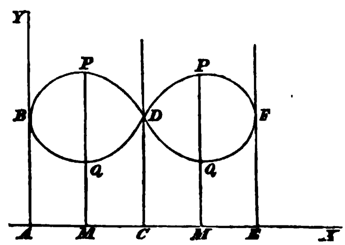
hence if AB and AD be made equal to 1 and 2 respectively, and the corresponding ordinates BC and DE be drawn, these ordinates will limit the curve in the direction of the axis of x:

and since $\dfrac{dy}{dx}=1\pm\dfrac{3-2x}{\sqrt{2(x-1)(2-x)}}$, which becomes infinite when the value of x is either 1 or 2, it follows that the ordinates BC and DE are tangents to the curve at the points C and E respectively.

Ex. 2. If the equation to the proposed curve be

$$\frac{y-a}{x-a} = \pm \frac{\sqrt{2ax-x^2}}{a},$$

which gives $y = a \pm \frac{x-a}{a}\sqrt{2ax-x^2}$, and the two values of y be made equal to each other, we find $x=0$, $x=a$ and $x=2a$: that is, if x be assumed equal to 0, a and $2a$ in succession, it is

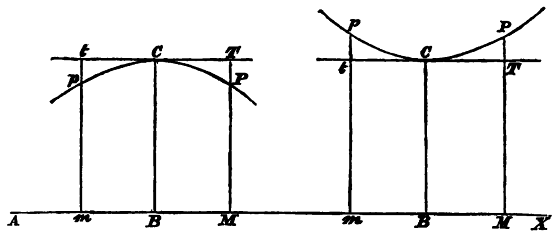


obvious that in each case the corresponding values of y are equal to one another and to a: also since the values of y become impossible when x is negative or greater than $2a$, it follows that the equal ordinates belonging to $x=0$ and $x=2a$ limit the curve in the direction of the axis of x, and are tangents to it at the points of concourse: whereas those corresponding to $x=a$, point out the intersection of two branches of the curve, a circumstance which will be more fully considered in some of the subsequent articles of the present Chapter.

(4) *Points corresponding to maximum and minimum Ordinates.*

217. If upon assigning any particular value to x, we find the value of $\frac{dy}{dx}$ the trigonometrical tangent of the angle which the rectilineal tangent makes with the axis of x to be 0, the tangent at the corresponding point in the curve is parallel to the axis of x, and the ordinate is a maximum or a minimum according as $\frac{d^2y}{dx^2}$ then becomes negative or positive.

These circumstances are evinced in the following diagrams:



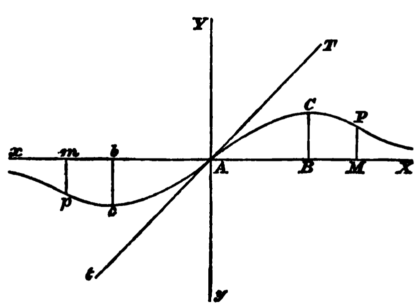
BC being the maximum and minimum ordinates in the first and second parts of the figure respectively, and the tangent tCT being in each case parallel to the axis of x. Similar conclusions may be deduced with reference to the axis of y by the consideration of the differential coefficients $\dfrac{dx}{dy}$ and $\dfrac{d^2x}{dy^2}$.

Hence if $u = f(x, y) = 0$ be the equation of a curve proposed, and

$$du = Pdx - Qdy = 0,$$

be derived from it by one differentiation, it is manifest that the equations $\dfrac{du}{dx} = 0$ and $\dfrac{du}{dy} = 0$, which are coexistent with $\dfrac{dy}{dx} = \dfrac{P}{Q} = 0$ and $\dfrac{dx}{dy} = \dfrac{Q}{P} = 0$ will determine the values of x and y which belong to the greatest and least values of y and x respectively.

Ex. 1. Let the equation to the curve be $y = \dfrac{x}{1+x^2}$;



then will $\frac{dy}{dx} = \frac{1-x^2}{(1+x^2)^2}$ and $\frac{d^2y}{dx^2} = \frac{2x(x^2-3)}{(1+x^2)^3}$;

now if x be assumed $= \pm 1$, we find corresponding thereto

$$y = \pm \frac{1}{1+1} = \pm \tfrac{1}{2} \text{ and } \frac{dy}{dx} = 0,$$

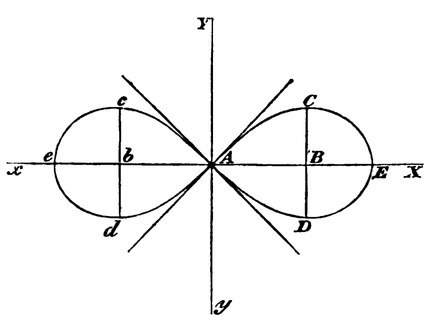
so that $y = \pm \frac{1}{2}$ is either a maximum or a minimum: but when $x = 1$, we have $\frac{d^2y}{dx^2} = -\frac{1}{2}$, and when $x = -1$, $\frac{d^2y}{dx^2} = \frac{1}{2}$: therefore if AB and Ab be each taken $= 1$, the corresponding ordinates BC and bc will be respectively a maximum and a minimum, the tangents at the points C and c being parallel to the axis of x.

The ordinate bc which is a minimum when estimated in the direction of AY will obviously be a maximum when it is estimated in the direction of Ay, as readily appears also from the criterion, if y in $\frac{d^2y}{dx^2}$ be considered negative.

Ex. 2. To determine the greatest and least values of the co-ordinates of the Lemniscata of *Bernoulli* whose equation is $(x^2+y^2)^2 = 2a^2(x^2-y^2)$, we have

$$u = x^4 + 2x^2y^2 + y^4 - 2a^2x^2 + 2a^2y^2 = 0:$$

whence the characteristic equations are readily obtained, that is,



$$\frac{du}{dx} = 4(x^3 + xy^2 - a^2x) = 0, \text{ and } \frac{du}{dy} = 4(y^3 + x^2y + a^2y) = 0:$$

from the former of these we have $x=0$ and $x^2+y^2=a^2$; and $x=0$ renders y impossible, whereas $x^2+y^2=a^2$ gives from the proposed equation

$$a^4=2a^2(x^2-y^2) \text{ or } x^2-y^2=\tfrac{1}{2}a^2:$$

whence are easily obtained $x=\pm\frac{1}{2}a\sqrt{3}$ and $y=\pm\frac{1}{2}a$;

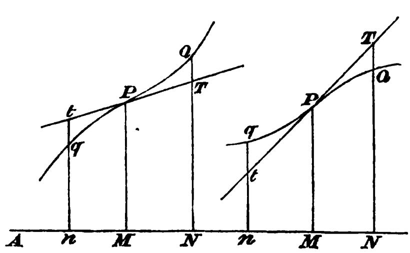
now $\dfrac{d^2y}{dx^2}=\dfrac{-(5x^2+y^2+a^2)(x^2+y^2)}{y(a^2+x^2+y^2)^2}$ when $\dfrac{dy}{dx}=0$, which is obviously a negative or positive quantity according as the value of y is positive or negative: that is, if AB and Ab be each made $=\frac{1}{2}a\sqrt{3}$, and the double ordinates CBD, cbd be drawn, BC, bc are positive maximum values of the ordinate, and BD, bd are negative maximum values of the same, and the tangents at the points C, c, D, d are parallel to the axis of x.

Similarly, $y(y^2+x^2+a^2)=0$ will enable us to determine the maximum and minimum values of x, which are AE and Ae in the diagram.

(5) *Points of Inflexion or contrary Flexure.*

218. If when any particular value is assigned to x, the value of the second differential coefficient of the ordinate $\dfrac{d^2y}{dx^2}$ become $=0$, the point of the curve in which it is intersected by the corresponding ordinate is generally a point of *Inflexion* or *contrary Flexure.*

For, let AP be a curve which on one side of P is concave towards the axis of x, and on the other convex, so that P



is a point of inflexion or contrary flexure: $AM=x$, $MP=y$, and $MN=Mn=h$: then it is evident that the deflections QT and qt from the rectilineal tangent tPT must have different algebraical signs, however small the quantity h may be assumed: now retaining the notation before adopted, we have

$$NT=y+ph,$$

$$\text{and } NQ=y+ph+q\frac{h^2}{1.2}+r\frac{h^3}{1.2.3}+\&c.,$$

$$\therefore\ NT-NQ=-\left(q\frac{h^2}{1.2}+r\frac{h^3}{1.2.3}+\&c.\right):$$

in like manner, we shall find $nt=y-ph$,

$$\text{and } nq=y-ph+q\frac{h^2}{1.2}-r\frac{h^3}{1.2.3}+\&c.,$$

$$\therefore\ nt-nq=-\left(q\frac{h^2}{1.2}-r\frac{h^3}{1.2.3}+\&c.\right):$$

wherefore if $q=0$, these differences become

$$NT-NQ=-r\frac{h^3}{1.2.3}-s\frac{h^4}{1.2.3.4}-\&c.$$

$$\text{and } nt-nq=r\frac{h^3}{1.2.3}-s\frac{h^4}{1.2.3.4}+\&c.;$$

which, since the first term of the series may by the diminution of h be made greater than the sum of all the rest, have different algebraical signs: and thus the curve after having in the former part of its course been situated above the tangent, will now be situated below it, or the contrary, and the roots of the equation $q=0$ will obviously be the values of the abscissa corresponding to the points of inflexion where this change takes place.

If p also $=0$, it is manifest that the tangent at the corresponding point of the curve becomes parallel to the axis of x, but the value of the ordinate is then neither a

maximum nor a minimum, since q is neither negative nor positive.

Ex. 1. Let the proposed curve be defined by the equation

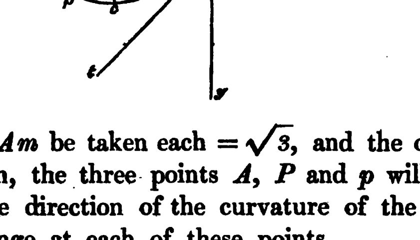
$$y = x + 36x^2 + 2x^3 - x^4:$$

$$\text{then } p = 1 + 72x + 6x^2 - 4x^3,$$

$$\text{and } q = 72 + 12x - 12x^2 = 12\,(6 + x - x^2):$$

whence at points of inflexion we have $6 + x - x^2 = 0$; the roots of which are 3 and -2: and therefore if the ordinates corresponding to these values of x be drawn, they will intersect the curve in points of contrary flexure.

Ex. 2. Let $y = \dfrac{x}{1+x^2}$ be the equation of the curve proposed: then we have $p = \dfrac{1-x^2}{(1+x^2)^2}$ and $q = \dfrac{2x^3 - 6x}{(1+x^2)^3}$, the latter of which being put $=0$, gives $x = 0$ and $x = \pm\sqrt{3}$: whence



if AM and Am be taken each $= \sqrt{3}$, and the ordinates MP, mp be drawn, the three points A, P and p will be points of inflexion, the direction of the curvature of the curve undergoing a change at each of these points.

Ex. 3. Let $x^2y + a^2y - ax^2 = 0$, or $y = \dfrac{ax^2}{a^2 + x^2}$: then we have

$$p = \frac{2a^3x}{(a^2+x^2)^2} \text{ and } q = \frac{2a^3(a^2-3x^2)}{(a^2+x^2)^3}:$$

$$\text{therefore } a^2 - 3x^2 = 0 \text{ gives } x = \pm\frac{a}{\sqrt{3}},$$

from which $y = \frac{1}{4}a$, so that $\frac{a}{\sqrt{3}}$, $\frac{a}{4}$ and $-\frac{a}{\sqrt{3}}$, $\frac{a}{4}$ are the co-ordinates of the points of inflexion, whose positions are therefore determined.

219. There are, however, some exceptions to the theory just explained, which it will now be expedient to notice: for if a value of x which satisfies the equation $q=0$, fulfil also the condition of the equation $r=0$, we shall have

$$NT - NQ = -s\frac{h^4}{1.2.3.4} - t\frac{h^5}{1.2.3.4.5} - \&c.$$

$$nt - nq = -s\frac{h^4}{1.2.3.4} + t\frac{h^5}{1.2.3.4.5} - \&c.;$$

and these, when h is assumed of a certain magnitude, will have the same algebraical sign, which circumstance, as above stated, cannot correspond to a point of inflexion: but if at the same time we have $s=0$, these equations become respectively

$$NT - NQ = -t\frac{h^5}{1.2.3.4.5} - \&c.$$

$$\text{and } nt - nq = t\frac{h^5}{1.2.3.4.5} - \&c.;$$

which, on the same supposition, have different algebraical signs, and consequently, these will correspond to a point of contrary flexure: and continuing to reason after the same manner, we may conclude generally that a curve has a point of inflexion or not according as the first differential coefficient that does not vanish, is of an odd or an even order.

Hence also, if the equation $q=0$ have m equal roots, there will obviously be a point of contrary flexure when m is odd, but none when m is even.

Ex. 1. Let $a^3y=(x-b)^4$, then we have $a^3p=4(x-b)^3$, $a^3q=12(x-b)^2$, $a^3r=24(x-b)$ and $a^3s=24$: but if q be made $=0$, we find $(x-b)^2=0$ or $x=b$, corresponding to which there is not a point of contrary flexure, since the *fourth* differential coefficient is the first that does not vanish: also if $b=0$, we have $x=0$, and there is no point of inflexion at the origin.

Ex. 2. If $a^4y=(x-b)^5$, we shall have $a^4p=5(x-b)^4$, $a^4q=20(x-b)^3$, $a^4r=60(x-b)^2$, $a^4s=120(x-b)$ and $a^4t=120$:

but if we make $q=0$ we find $x=b$; that is, when $x=b$, and consequently $y=0$, there is a point of inflexion, because the first differential coefficient that does not vanish is the *fifth*.

220. From what has already been said, it appears that the sole characteristic of a point of inflexion in a curve is a change of the algebraical sign of the second or other succeeding differential coefficient of an even order from $+$ to $-$ or from $-$ to $+$: and since, though an integral quantity can change its sign only in passing through zero, a fraction may do so either by passing through zero or infinity, it follows that all the points of contrary flexure belonging to any curve may be determined from one or both of the equations $q=0$ and $q=\infty$.

Ex. 1. Let $y=a^{\frac{2}{3}}x+(x-b)^{\frac{5}{3}}$, then $p=a^{\frac{2}{3}}+\frac{5}{3}(x-b)^{\frac{2}{3}}$

and $q=\dfrac{10}{9(x-b)^{\frac{1}{3}}}$: wherefore at a point of inflexion we can only

make $\dfrac{10}{9(x-b)^{\frac{1}{3}}}=\infty$, or $x=b$: also, if for x we put $b+h$, the

value of $q = \frac{10}{9h^{\frac{1}{3}}}$, and if $x = b - h$, the value of $q = -\frac{10}{9h^{\frac{1}{3}}}$; wherefore if $x = b$, and therefore $y = a^{\frac{2}{3}}b$, the corresponding point in the curve possesses the characteristic property of a point of inflexion.

Ex. 2. Let $ax^3 + by^3 + c^4 = 0$, then

$$p = -\frac{ax^2}{by^2} \text{ and } q = -\frac{2ax}{b^2}\left\{\frac{ax^3 + by^3}{y^5}\right\}:$$

whence if $q = 0$, we have $x = 0$ and $ax^3 + by^3 = 0$, the latter of which cannot take place since $ax^3 + by^3 + c^4 = 0$: therefore when $x = 0$ and $y = -\frac{c^{\frac{4}{3}}}{b^{\frac{1}{3}}}$, there is a point of contrary flexure: also, if $q = \infty$, we find $y = 0$ and $x = -\frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}}$, that is, $x = -\frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}}$ and $y = 0$ belong to a point of contrary flexure.

221. If x be considered the dependent, and y the independent variable, the principles already developed will lead us to the conclusions that at points of inflexion we must have $\frac{d^2x}{dy^2} = 0$ or $\frac{d^2x}{dy^2} = \infty$: and examples in which x is found to be an explicit function of y will be more easily solved by means of one or both of these expressions.

Ex. The equation to the Conchoid of *Nicomedes* is

$$x = (ay^{-1} + 1)\sqrt{b^2 - y^2}:$$

$$\text{whence } \frac{dx}{dy} = -\frac{ab^2 + y^3}{y^2\sqrt{b^2 - y^2}},$$

$$\text{and } \frac{d^2x}{dy^2} = \frac{b^2(y^3+3ay^2-2ab^2)}{y^3(b^2-y^2)^{\frac{3}{2}}};$$

the latter of which being put $=0$, gives $y^3+3ay^2-2ab^2=0$, from which the values of y may be found, and thence those of x by means of the proposed equation.

If $b=a$, this last equation becomes

$$y^3+3ay^2-2a^3=(y+a)(y^2+2ay-2a^2)=0,$$

whose roots are $-a$, $a(\sqrt{3}-1)$ and $-a(\sqrt{3}+1)$, for the first two of which alone the values of x are possible, and the last does not belong to any point in the curve.

222. Cor. 1. From an examination of the circumstances of a point of inflexion geometrically exhibited, it will be obvious that the trigonometrical tangent expressed by $\frac{dy}{dx}$ or $\frac{dx}{dy}$ must then be either a maximum or a minimum, from which we infer also that $\frac{d^2y}{dx^2}=0$ or ∞, and $\frac{d^2x}{dy^2}=0$ or ∞, unless $\frac{dy}{dx}$ or $\frac{dx}{dy}$ exhibit some peculiarity either in their form or value.

223. Cor. 2. We have seen in (176) of the last Chapter but one that

$$\gamma = \frac{(1+p^2)^{\frac{3}{2}}}{q};$$

whence it follows that if p be not infinite and q be made equal to zero or infinity, the radius of curvature at a point of inflexion will be either infinite or evanescent.

224. Before we quit this sub-division of the present Chapter, it may not be inexpedient to introduce two or three additional examples connected with Inflexion, in order more particularly to explain and illustrate certain terms made use of

by some writers upon the subject, and which, though immaterial in a practical point of view, nevertheless evince the power of Analysis when applied to Geometry.

Ex. 1. Let the curve be defined by the equation

$$a^2y = x^3 - cx^2 - b^2x + b^2c,$$

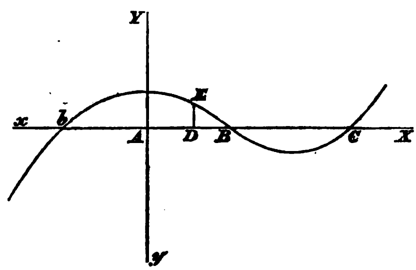
which admits also of the form

$$a^2y = (x+b)(x-b)(x-c):$$

then we have $a^2p = 3x^2 - 2cx - b^2$,

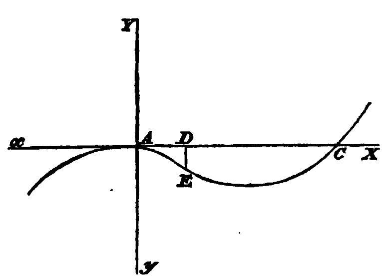
$$a^2q = 6x - 2c, \quad a^2r = 6, \quad a^2s = 0:$$

wherefore if $q = 0$, we obtain $x = \frac{1}{3}c$, corresponding to which there exists a point of inflexion as at E in the following diagram, where A is the origin, and xAX and YAy the axes



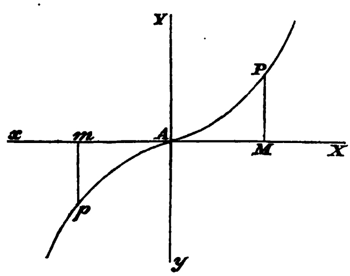
of co-ordinates, $AB = Ab = b$, $AC = c$ and $AD = \frac{1}{3}c$.

If $b = 0$, the two points of intersection B and b unite



in one at the origin A: and AD being $= \frac{1}{3}c$, the point of inflexion will be at E.

If we have $b=0$ and $c=0$, so that the equation becomes $a^2y=x^3$, the three points B, C and b all unite at the origin, and so does the point of inflexion E, as at the point A below.



In the latter case, the origin of co-ordinates is a point of *Single Inflexion*, though the union of three points of the curve takes place there, the existence of which is indeed indicated by the equation

$$a^2y = x^3 = x \cdot x \cdot x,$$

but Geometry is not possessed of the power to exhibit them.

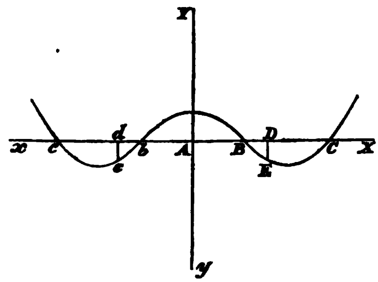
Ex. 2. Let the equation of the curve proposed be

$$\begin{aligned} a^3y &= x^4 - (b^2+c^2)\,x^2 + b^2c^2 \\ &= (x+b)\,(x-b)\,(x+c)\,(x-c): \end{aligned}$$

$$\begin{aligned} \therefore\ a^3p &= 4x^3 - 2\,(b^2+c^2)\,x, \\ a^3q &= 12x^2 - 2\,(b^2+c^2), \\ a^3r &= 24x,\ \ a^3s = 24,\ \ a^3t = 0; \end{aligned}$$

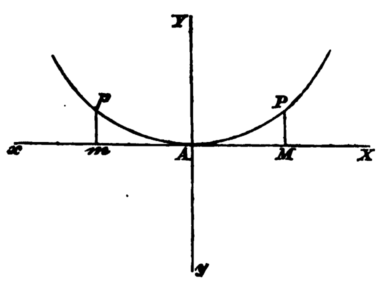
whence if $q=0$, we obtain $x = \pm\sqrt{\dfrac{b^2+c^2}{6}}$, which, because then $r = \pm\dfrac{24}{a^3}\sqrt{\dfrac{b^2+c^2}{6}}$, indicate two points of contrary

flexure, as at the points E and e in the following diagram:



that is, if $AB=Ab=b$, $AC=Ac=c$, the curve intersects the axis of x in B, b, C, c, and making $AD=Ad=\sqrt{\dfrac{b^2+c^2}{6}}$, we have the two points of inflexion at E and e.

Hence if b and c be each assumed $=0$, the four points of intersection B, b, C, c, as well as the two points of contrary flexure E, e will all be united in one at the origin, which will therefore become a point of *Double Inflexion*, and may be represented by A in the following diagram, the double in-



flexion there not being visible as was the single inflexion in the last example.

From the form which the curve retains whilst the quantities b and c remain finite, the origin of the co-ordinates of the curve whose equation is $a^3y=x^4=x.x.x.x$ has been designated a point of *Undulation*; and the term *Serpentement*

has been applied to a point thus circumstanced by some of the French mathematicians. The existence of such a kind of point indeed depends entirely upon the supposition of the quantities b and c having been previously regarded as finite magnitudes, for according to the ordinary principles of inflexion above explained, we have

$$a^3p = 4x^3, \quad a^3q = 12x^2, \quad a^3r = 24x, \quad a^3s = 24,$$

from which if $q = 0$, we find $x = 0$; but this does not here correspond to a point of contrary flexure, because the first differential which this value of x causes not to vanish is s, which is of an even order: that is, there is naturally no inflexion at the origin, and the point of double inflexion there originates solely from the union of the two points determined on the afore-mentioned hypothesis.

Ex. 3. If $a^4y = x^5 - (b^2 + c^2)\, x^3 + b^2c^2x$

$$= x\,(x + b)\,(x - b)\,(x + c)\,(x - c),$$

we shall have

$$a^4p = 5x^4 - 3\,(b^2 + c^2)\, x^2 + b^2c^2,$$

$$a^4q = 20x^3 - 6\,(b^2 + c^2)\, x,$$

$$a^4r = 60x^2 - 6\,(b^2 + c^2):$$

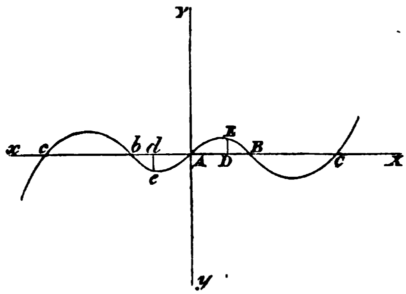
whence assuming $q = 0$, we obtain

$$20x^3 - 6\,(b^2 + c^2)\, x = 0;$$

that is, $x\,\{10x^2 - 3\,(b^2 + c^2)\} = 0$, the roots of which are obviously 0 and $\pm\sqrt{\dfrac{3\,(b^2 + c^2)}{10}}$ correspondent to points of contrary flexure, one of which is the origin of co-ordinates; thus, if we make $AB = Ab = b$, $AC = Ac = c$ and

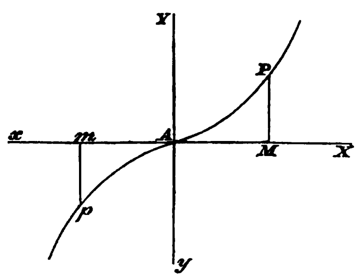
$$AD = Ad = \sqrt{\frac{3\,(b^2 + c^2)}{10}},$$

and draw the ordinates DE, de, the four points B, b, C, c,



will be points of intersection of the axis of x whilst A, E, e are the three points of inflexion.

In this case, the quantities b and c being supposed to vanish, the two points E and e coincide with the point A, and thus the origin becomes a point of *Triple Inflexion* whose geometrical character is similar to that of a point of Simple Inflexion, as in the following diagram.



Here the equation $a^4 y = x^5$ gives

$$a^4 p = 5x^4,\ a^4 q = 20x^3,\ a^4 r = 60x^2,\ a^4 s = 120x,\ a^4 t = 120;$$

so that the fifth differential coefficient is the first which does not vanish when $x = 0$, and the ordinary principles point out merely a simple inflexion at the origin, the triple order of inflexion depending for its existence upon considerations similar to those mentioned in the last example.

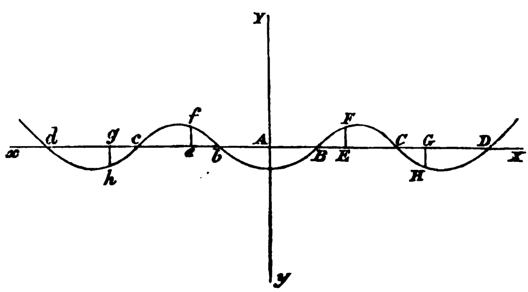
Ex. 4. Taking the parabolic curve whose equation is

$$y = x^6 - (a^2 + b^2 + c^2)\,x^4 + (a^2b^2 + a^2c^2 + b^2c^2)\,x^2 - a^2b^2c^2$$
$$= (x^2 - a^2)\,(x^2 - b^2)\,(x^2 - c^2),$$

we shall have

$$p = \quad 6x^5 - \;\;4\,(a^2 + b^2 + c^2)\,x^3 + 2\,(a^2b^2 + a^2c^2 + b^2c^2)\,x,$$
$$q = \;30x^4 - 12\,(a^2 + b^2 + c^2)\,x^2 + 2\,(a^2b^2 + a^2c^2 + b^2c^2),$$
$$r = 120x^3 - 24\,(a^2 + b^2 + c^2)\,x\,;$$

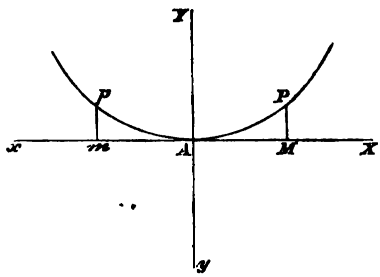
and from the equation $q = 0$, which is characteristic of inflexion, four different values of x will be obtained which manifestly belong to so many different points of contrary flexure: thus, in the following figure, if



$AB = Ab = a$, $AC = Ac = b$, $AD = Ad = c$, and AE, AG, Ae, Ag be taken equal to the roots of the equation $q = 0$, the corresponding points F, H, f, h will be points of contrary flexure according to the principles above laid down.

If each of the magnitudes a, b, c be supposed $= 0$, it is obvious that the four points of inflexion F, H, f, h unite in one at the origin of the co-ordinates and form what is called a point of *Quadruple Inflexion* there, though the figure will not then possess the appearance of any inflexion at all, as in the following diagram, nor will the equation $y = x^6$ by the ordinary principles lead to the discovery of its existence, as in Example 2. A point supposed thus to arise from the

union of four points of contrary flexure in a curve, is desig-



nated a point of *Double Undulation*, and a similar mode of reasoning may obviously be applied to curves defined by equations of higher dimensions.

225. Cor. From the discussions of the preceding examples, we may therefore collect that all points of odd orders of inflexion originating as above described, agree in their geometrical character with points of simple inflexion, and are therefore visible; but that all points of even orders of contrary flexure possess no visible geometrical character whatever, and are therefore to be interpreted exclusively in an analytical sense as having reference to the pre-existence of certain real magnitudes, which in the case under immediate consideration have become evanescent.

And if the equation $q=0$ have two equal roots, we are led to infer that the corresponding point in the curve is one of double inflexion: if it have three equal roots, the curve has a point of triple inflexion to correspond, and so on.

(6) *Isolated or Conjugate Points.*

226. If when a certain value is given to the abscissa x, the value of the ordinate y remain possible at the same time that one or more of its differential coefficients become imaginary, the corresponding point of the curve is an *Isolated* or *Conjugate Point*.

For, if the equation to the curve be $y=f(x)$, then by *Taylor's* Theorem we have

$$y'=f(x+h)=y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\&c.$$

$$y_{,}=f(x-h)=y-\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}-\&c.,$$

which will both manifestly become impossible whatever h may be assumed when such a value is assigned to the principal variable, as shall render any of the differential coefficients imaginary. If, therefore, two possible co-ordinates render any of the differential coefficients impossible, the point defined by the said co-ordinates being detached from the other parts of the curve will be what is called an isolated or conjugate point.

Hence it is manifest that to determine the positions of such points, we have only to ascertain what possible values of the co-ordinates satisfy the equation of the curve $y=f(x)$ at the same time that they render one or more of the differential coefficients imaginary.

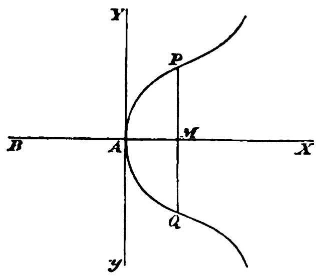
227. The impossibility of the differential coefficients may manifestly arise from the indication of the extraction of an even root of a negative quantity when the equation to the curve involves surds, as will appear by the following instance.

Ex. In the curve whose equation is $ay^2=x(x+b)^2$, we have immediately $y=\pm(x+b)\sqrt{\frac{x}{a}}$, and

$$\frac{dy}{dx}=\pm\frac{1}{2\sqrt{a}}\left\{3\sqrt{x}+\frac{b}{\sqrt{x}}\right\}:$$

now if $x=-b$, we have $y=0$, and the corresponding value of $\frac{dy}{dx}=\mp\sqrt{-\frac{b}{a}}$ will cause the values preceding and succeed-

ing this to be imaginary, so that if A be the origin of the



co-ordinates, and AB be taken $= b$, the point B belongs to the curve, but is entirely detached from the rest of the figure, assuming the character of a conjugate point.

228. The impossibility of one or more of the differential coefficients, and consequent existence of a conjugate point, may be indicated by the circumstance of the first differential coefficient appearing in the particular case under the indeterminate form $\frac{0}{0}$, when the equation is free from surds.

For, let the first differentiation of the equation of the curve give

$$Pdx + Qdy = 0, \quad \text{or } P + Q\frac{dy}{dx} = 0,$$

where P and Q are functions of x and y: and this by successive repetitions of the same operation obviously leads to a result of the form

$$Rdx^{n+1} + Qd^{n+1}y = 0, \quad \text{or } R + Q\frac{d^{n+1}y}{dx^{n+1}} = 0:$$

now if the differential coefficient $\frac{d^{n+1}y}{dx^{n+1}}$ become imaginary, it is manifest that we must have R and Q independently equal

to zero, that is $R=0$ and $Q=0$; and the equation $P+Q\frac{dy}{dx}=0$ leads immediately to $P=0$, so that $\frac{dy}{dx}=\frac{0}{0}$.

Ex. Let the curve be defined by the equation $x^2y^2=(a^2-x^2)(x-2a)^2$, from which is immediately deduced

$$\frac{dy}{dx}=\frac{(x-2a)(a^2+2ax-2x^2)}{xy(x+y)}:$$

now if the value $2a$ be assigned to x, we shall have $y=0$, and the first differential coefficient $\frac{dy}{dx}$ assumes the form $\frac{0}{0}$: but by (107) we find

$$p=-\frac{6x^2+4ax+3a^2}{2xy+y^2+(x^2+2xy)p}:$$

$$\therefore\ (2xy+y^2)p+(x^2+2xy)p^2=-(6x^2+4ax+3a^2):$$

and if $x=2a$ and $y=0$, we have

$$4a^2p^2=-35a^2 \text{ or } p=\pm\tfrac{1}{2}\sqrt{-35},$$

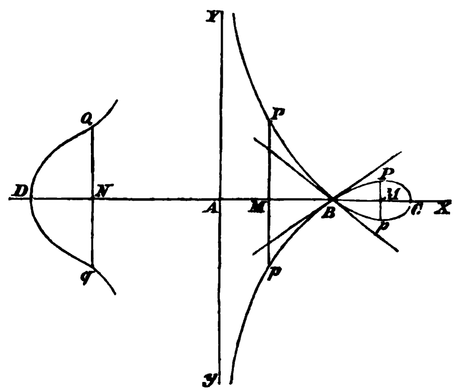
which is an imaginary quantity: hence the ordinates expressed generally by $y=f(x\pm h)$ have no existence contiguous to the point whose co-ordinates are $2a$ and 0, which is therefore a detached or conjugate point.

The curve just considered is a particular case of the more general one whose equation is

$$x^2y^2=(a^2-x^2)(x-b)^2,$$

and is delineated underneath, the point C being beyond the point B in consequence of b being greater than a, and having become a conjugate point: and it may be observed that the existence of a conjugate point is generally attributable to the pre-existence of some *Node* or *Oval*, the magnitude of

which depended upon the value of a constant in the equation,



and in the particular case the magnitude of the said constant is such that the oval or node reduces itself to a point.

(7) *Points of Maximum and Minimum Curvature.*

229. We have seen in (176) that the radius of curvature of a curve is expressed generally in the formula

$$\gamma = \frac{(1+p^2)^{\frac{3}{2}}}{q};$$

whence it follows that when such a value is assigned to x as renders $\frac{d\gamma}{dx} = 0$ or ∞, the corresponding point in the curve will have its radius of curvature a maximum or a minimum according as $\frac{d^2\gamma}{dx^2}$ is negative or positive, and so will be a point of minimum or maximum curvature agreeably to what is said in (175).

Ex. In the rectangular hyperbola referred to the asymptotes, we have seen in Ex. 3. of (177) that

$$\gamma = \frac{1}{2a^2}\left(x^2 + \frac{a^4}{x^2}\right)^{\frac{3}{2}};$$

the maximum or minimum value of which is determined from

$$\frac{d\gamma}{dx} = \frac{3}{2a^2}\left(x - \frac{a^4}{x^3}\right)\left(x^2 + \frac{a^4}{x^2}\right)^{\frac{1}{2}} = 0;$$

and the only possible solution of this gives $x = \pm a$, which, as might easily be shewn, renders $\frac{d^2\gamma}{dx^2}$ a positive quantity: at this point, therefore, the radius of curvature is a minimum, and consequently the curvature itself is a maximum at each of the vertices of the hyperbola.

230. When the contact subsisting between a curve and its circle of curvature is of an odd order, the curvature is either a maximum or a minimum, but when it is of an even order, the curvature is neither.

For, since $\gamma = \dfrac{\left(1 + \dfrac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}} = \dfrac{\left(1 + \dfrac{dy'^2}{dx'^2}\right)^{\frac{3}{2}}}{\dfrac{d^2y'}{dx'^2}}$,

if we call the successive differential coefficients of the ordinate of the curve and of the circle of curvature p, q, r, s, &c., p', q', r', s', &c. respectively, and take notice that by each successive differentiation we introduce an additional differential coefficient, we shall manifestly have for the curve

$$\frac{d\gamma}{dx} = \phi\,(p, q, r), \frac{d^2\gamma}{dx^2} = \phi_1\,(p, q, r, s), \frac{d^3\gamma}{dx^3} = \phi_2\,(p, q, r, s, t), \text{\&c.},$$

and for the circle whose radius undergoes no corresponding change,

$$0 = \phi\,(p', q', r'),\ 0 = \phi_1\,(p', q', r', s'),\ 0 = \phi_2\,(p', q', r', s', t'),\ \text{\&c.}:$$

whence, if $p' = p$, $q' = q$ and $r' = r$, we shall have

$$\frac{d\gamma}{dx} = \phi\,(p, q, r) = \phi\,(p', q', r') = 0;$$

and therefore γ is a maximum or a minimum according as

$\dfrac{d^2\gamma}{dx^2}$ is negative or positive: but if, in addition to these we have likewise $s'=s$, but not $t'=t$, it follows that

$$\frac{d^2\gamma}{dx^2}=\phi_1(p, q, r, s)=\phi_1(p', q', r', s')=0,$$

and therefore γ can be neither a maximum nor a minimum: and similarly of succeeding orders.

Hence also, conversely, if at any point of a curve the curvature be a maximum or a minimum, the contact of the circle of curvature with it at that point is of an odd order.

Ex. 1. In an Ellipse, whose co-ordinates are measured from the centre, it has been proved in Ex. 2. of (177) that

$$\gamma=-\frac{\{a^4-(a^2-b^2)x^2\}^{\frac{3}{2}}}{a^4 b};$$

$$\therefore \frac{d\gamma}{dx}=\frac{3(a^2-b^2)x}{a^4 b}\{a^4-(a^2-b^2)x^2\}^{\frac{1}{2}}=0,$$

when γ is either a maximum or a minimum;

$$\text{whence we have } x=0 \text{ and } x=\pm\frac{a^2}{\sqrt{a^2-b^2}}:$$

and the former of these is proved by the usual criterion to correspond to a maximum, and the latter neither to a maximum nor a minimum:

$$\text{now } p=-\frac{b}{a}\frac{x}{\sqrt{a^2-x^2}},\ q=-\frac{ab}{(a^2-x^2)^{\frac{3}{2}}},$$

$$\text{and } r=-\frac{3abx}{(a^2-x^2)^{\frac{5}{2}}},\text{ which }=0,\text{ when } x=0:$$

also, from the equation $\gamma^2=(x'-\alpha)^2+(y'-\beta)^2$, we find, by differentiation, substitution and elimination,

$$r' = \frac{3pq^2}{1+p^2},$$

which in this case becomes

$$= -\frac{3a^3b^3x}{(a^2-x^2)^{\frac{5}{2}}\{a^4-(a^2-b^2)x^2\}} = 0;$$

whence $r'=r$, or the ellipse and its circle of curvature at the extremities of the minor axis have contact of the third order, the radius of curvature being there a maximum, and consequently the curvature itself a minimum.

Also, since at the point whose co-ordinates are 0 and b,

$$\text{we have } s = -\frac{3b}{a^4} \text{ and } s' = -\frac{3b^3}{a^6},$$

the curve and its circle of curvature have not contact of a higher order than the third; and because $s'-s = \frac{3b}{a^6}(a^2-b^2)$ is a positive quantity, it follows that the circle of curvature, at the extremity of the minor axis, falls without the curve.

Similarly, if we take $\gamma = -\frac{\{b^4+(a^2-b^2)y^2\}^{\frac{3}{2}}}{ab^4}$, we shall find $y=0$, or $x=a$, when γ is a minimum, and $r'=r$, but s' less than s; and therefore the circle of curvature, at the extremity of the major axis, falls within the ellipse.

Ex. 2. In the cubical Parabola $y^3=a^2x$: whence by (176)

$$\gamma = -\frac{(a^{\frac{4}{3}}+9x^{\frac{4}{3}})^{\frac{3}{2}}}{6a^{\frac{2}{3}}x^{\frac{1}{3}}};$$

and if this be a minimum, we find $x = \pm\frac{a}{(45)^{\frac{3}{4}}}$: wherefore $r = \frac{750}{a^2} = r'$, or the curve and its circle of curvature have con-

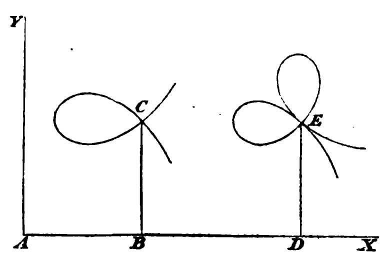
tact of the third order where the curvature is a maximum; and since s' is at the same point less than s, the circle of curvature falls within the curve.

Points possessing the analytical characters exhibited in these examples form the exceptions in particular cases alluded to in Article (163).

MULTIPLE POINTS.

(1) *Points in which two or more Branches intersect each other.*

231. If when a certain value is assigned to the abscissa x, the ordinate y have only one value, but its first differential coefficient $\frac{dy}{dx}$ admits of more than one, it is obvious that the curve admits of more rectilineal tangents than one at the corresponding point, which is therefore designated a multiple point whose degree of multiplicity is expressed by the number of the said real values of $\frac{dy}{dx}$: thus, in the first and second parts of the following diagram, the points C and E are respectively



a *double* and a *triple* point; and the determination of such points will manifestly be effected by finding the magnitudes of the co-ordinates which give to $\frac{dy}{dx}$ more values than one.

232. Different values of $\frac{dy}{dx}$ corresponding to one value of each of the co-ordinates x and y may obviously originate

from the circumstance of a radical quantity, which has disappeared in the particular value of y, making its re-appearance in the first differential coefficient $\frac{dy}{dx}$, as will be seen in the following instances.

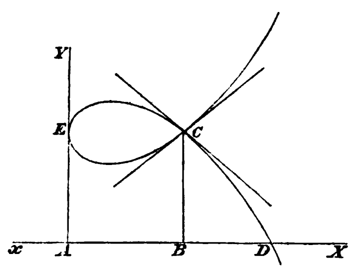
Ex. 1. Let the equation of the curve proposed be

$$y-b=\pm(x-a)\sqrt{x}:$$

then if to x be assigned the value a, it is obvious that we shall have two values of y each equal to b, by the disappearance of the radical quantity $\sqrt{x}$: but from this equation we obtain

$$\frac{dy}{dx}=\pm\frac{3x-a}{2\sqrt{x}},$$

which, when x is assumed $=a$, shews the values of the trigonometrical tangents of the angles at which the curve is there



inclined to the axis of x to be $\pm\sqrt{a}$: that is, A being the origin of co-ordinates, we have corresponding to $x=a=AB$ and $y=b=BC$, the double point C at which the positions of the rectilineal tangents are determined from the equation

$$\tan X=\pm\sqrt{a}.$$

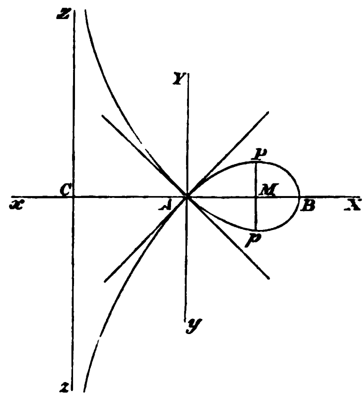
Ex. 2. If $(a+x)y^2=(a-x)x^2$, which gives for y the two general values $\pm x\sqrt{\frac{a-x}{a+x}}$ obviously uniting in one at the origin where $x=0$, we shall have

$$\frac{dy}{dx} = \pm \frac{(a^2 - ax - x^2)}{(a+x)\sqrt{a^2 - x^2}}:$$

but corresponding to $x = 0$, this expression admits of the two values ± 1, so that at this point we have

$$\tan X = \pm 1 = \tan 45^0 \text{ or } \tan 135^0;$$

hence at the point A in the following diagram, the curve admits



of two rectilineal tangents, making angles of 45^0 and 135^0 respectively with the axis of x, and has consequently a double point there.

Ex. 3. If $x^2y^2 = (a^2 - x^2)(x-b)^2$, we shall have

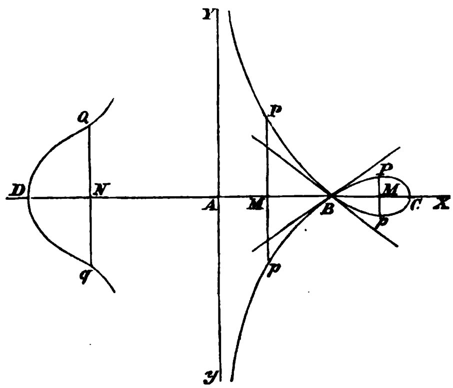
$$y = \pm \frac{x-b}{x}\sqrt{a^2 - x^2};$$

from which is immediately obtained

$$\frac{dy}{dx} = \pm \frac{a^2 b - x^3}{x^2\sqrt{a^2 - x^2}}:$$

now when $x = b$, it is obvious that each of the values of y becomes $= 0$, so that two points of the curve there coincide: but the values of $\frac{dy}{dx}$ on the same assumption manifestly become $= \pm \frac{\sqrt{a^2 - b^2}}{b}$, indicating the trigonometrical tangents

of the angles which the two branches of the curve there make with the axis of x, as is exhibited at the point B in the diagram underneath.



233. Whenever the differential coefficient $\frac{dy}{dx}$ appears in a form divested of radical quantities, it is evident that for one value of x and two or more equal values of y, it cannot generally have more than one value to correspond, except it be expressed in the form of a rational fraction, which is an implicit function of x and y, and assume the indeterminate form $\frac{0}{0}$, as appears from the instances given in (107).

Indeed it is easily demonstrated, that if the equation of a curve be freed from radical quantities, the differential coefficient $\frac{dy}{dx}$ corresponding to a multiple point, to which belong more than one rectilineal tangent, must generally assume the form $\frac{0}{0}$.

For, let $Pdx - Qdy = 0$ be the differential equation of the curve, the quantities P and Q being rational functions of the co-ordinates x and y, and suppose α and β to represent the trigonometrical tangents of the angles of inclination to the

axis of x at a double point: then the equation $P-Q\frac{dy}{dx}=0$, for this point gives

$$P-Q\alpha=0 \text{ and } P-Q\beta=0:$$

whence we have immediately

$$Q(\alpha-\beta)=0;$$

and this leads to the conclusion that $Q=0$ and $P=0$, so that $\frac{dy}{dx}$ which is represented generally by the rational fraction $\frac{P}{Q}$ assumes in this particular instance the form $\frac{0}{0}$.

It will not follow conversely however, that the values of the co-ordinates which give to $\frac{dy}{dx}$ this peculiar form, belong to a multiple point where two or more branches of the curve intersect each other, unless it admit of two or more different magnitudes when its true values are found by means of the principles laid down in (107.)

234. *To determine the multiple Points of a curve formed by the Intersections of its different Branches.*

Let the equation of the curve be expressed by $u=0$, where u is a function of its co-ordinates x and y divested of irrational quantities: then if the first differentiation give rise to the equation

$$Pdx-Qdy=0,$$

it is obvious that P and Q are the differential coefficients of u with reference to the mutually dependent variables x and y respectively: that is, $P=\frac{du}{dx}$ and $Q=\frac{du}{dy}$:

since, therefore, at the kind of point we are seeking, we must have $\frac{P}{Q}=\frac{0}{0}$, in order to determine the positions of such

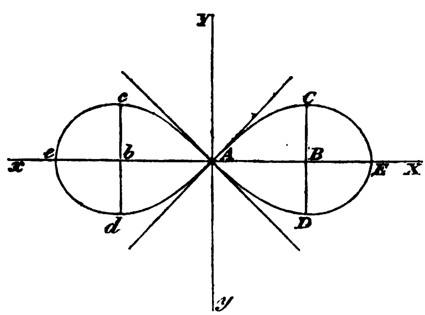
multiple points, the values of the co-ordinates must be so assumed as simultaneously to satisfy the three equations

$$u=0,\ \frac{du}{dx}=0 \text{ and } \frac{du}{dy}=0:$$

and to ascertain their nature and degree of multiplicity, all the corresponding values of $\frac{dy}{dx}$ must be found as the article (107) directs.

Ex. 1. The equation to the Lemniscata of *Bernoulli* being $(x^2+y^2)^2=2a^2(x^2-y^2)$, we obtain

$$\frac{dy}{dx}=\frac{a^2x-x^3-xy^2}{a^2y+y^3+x^2y}:$$



therefore at a multiple point we must have

$$u=(x^2+y^2)^2-2a^2(x^2-y^2)=0,$$

$$\frac{du}{dx}=a^2x-x^3-xy^2=0,$$

$$\frac{du}{dy}=a^2y+y^3+x^2y=0:$$

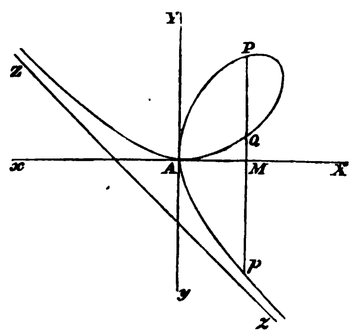
now it is obvious that when $x=0$, two values of y are equal to 0, and all the equations $u=0$, $\frac{du}{dx}=0$ and $\frac{du}{dy}=0$ are satis-

fied: that is, the origin of the co-ordinates may be a multiple point belonging to which we have by (107)

$$p = \frac{a^2 - 3x^2 - y^2 - 2pxy}{(a^2 + x^2 + 3y^2)\,p + 2xy}$$

wherein $x=0$ and $y=0$: whence we find $p = \frac{1}{p}$ or $p = \pm 1$; and therefore the curve has at the origin a double point, the rectilineal tangents to the two branches forming it being inclined to the axis of x at the respective angles of 45^0 and 135^0 as in the diagram.

Ex. 2. Let $y^3 - 3axy + x^3 = 0$, which gives $\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$:



then we have the three following equations

$$u = y^3 - 3axy + x^3 = 0,$$

$$\frac{du}{dx} = ay - x^2 = 0,$$

$$\frac{du}{dy} = y^2 - ax = 0,$$

to characterize the existence of a multiple point;

from $ay - x^2 = 0$ and $y^2 - ax = 0$, we have by addition

$$y^2 - x^2 + a(y - x) = 0, \text{ or } (y - x)(y + x + a) = 0:$$

$$\therefore\ y = x \text{ or } y = -(a + x):$$

of these $y=x$ by substitution in the proposed equation gives $x=0$, and $x=\frac{3}{2}a$, the first of which alone satisfies the equations $\frac{du}{dx}=0$ and $\frac{du}{dy}=0$, and $y=-(a+x)$ gives $a^3=0$, which is absurd:

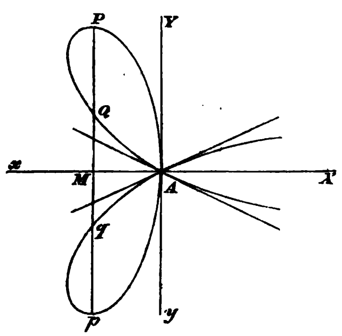
hence the origin of the co-ordinates may here be a multiple point:

$$\text{and because } p=\frac{ay-x^2}{y^2-ax}=\frac{0}{0}=\frac{ap-2x}{2yp-a}, \text{ by (107)},$$

the values of $\frac{dy}{dx}$ at the origin are 0 and ∞, which shew that this point of the curve is a double one of such a nature that the co-ordinate axes are tangents to its different branches, as in the diagram presented above.

Ex. 3. Let the curve be defined by the equation

$$y^4+2axy^2-ax^3=0: \text{ then } p=\frac{3ax^2-2ay^2}{4y^3+4axy}:$$



whence corresponding to multiple points, we must have

$$y^4+2axy^2-ax^3=0,$$
$$3ax^2-2ay^2=0,$$
$$4y^3+4axy=0,$$

which can be simultaneously satisfied only by the values $x=0$ and $y=0$:

$$\text{also, } p=\frac{3ax^2-2ay^2}{4y^3+4axy}=\frac{0}{0}=\frac{3ax-2ayp}{6y^2p+2ay+2axp}=\frac{0}{0}$$

$$=\frac{3a-2ap^2}{12yp^2+4ap} \text{ by (107)}:$$

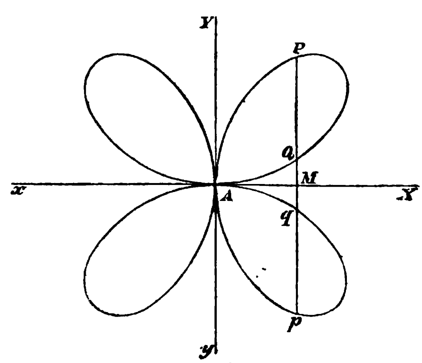
therefore $12yp^3+6ap^2-3a=0$, which, when $y=0$, becomes

$$6ap^2-3a=0, \text{ and gives } p=\pm\frac{1}{\sqrt{2}};$$

and it is moreover obvious that this equation will be satisfied by $p=\infty$:

hence in this case the origin is a triple point, to one branch of which the axis of y is a tangent, and the two others are touched by the straight lines, the trigonometrical tangents of whose inclinations to the axis of x are $\frac{1}{\sqrt{2}}$ and $-\frac{1}{\sqrt{2}}$, as in the diagram above given.

Ex. 4. Let the equation to the curve be $(x^2+y^2)^3-4a^2x^2y^2=0$, from which $\frac{dy}{dx}=-\frac{4a^2xy^2-3x(x^2+y^2)^2}{4a^2x^2y-3y(x^2+y^2)^2}$:



now it is readily seen that $x=0$ and $y=0$, are the only values which satisfy at the same time the three equations

$$(x^2+y^2)^3-4a^2x^2y^2=0,$$
$$4a^2xy^2-3x(x^2+y^2)^2=0,$$
$$4a^2x^2y-3y(x^2+y^2)^2=0:$$

and the determination of the corresponding values of $\frac{dy}{dx}$ according to the principles laid down in (107), shews that the origin is a quadruple point, the co-ordinate axes being tangents to the different branches of the curve as exhibited in the diagram above affixed.

(2) *Points in which two or more Branches touch each other.*

235. By assigning to the co-ordinates x and y particular values, the first differential coefficient $\frac{dy}{dx}$ may however assume the indeterminate form $\frac{0}{0}$, without at the same time admitting of more than one magnitude. This circumstance generally indicates the existence of points of contact of different branches of the curve as may be demonstrated by the following converse operation.

For, let two branches of the curve have with each other contact of the n^{th} order, so that the first n differential coefficients of y shall be the same for each branch of the curve, and those succeeding different: then from the equation $Pdx+Qdy=0$, we obtain, by n successive differentiations, the result $R\,dx^{n+1}+Qd^{n+1}y=0$, wherein R is a rational function of x, y and the first n successive differential coefficients, and Q the same as before:

now let the two values of $\frac{d^{n+1}y}{dx^{n+1}}$ which belong to the proposed point of contact be α and β so that we have $R+Q\alpha=0$

and $R+Q\beta=0$; then as before we conclude immediately that $Q=0$ and $R=0$, whence it follows also that $P=0$, and thence

$$\frac{dy}{dx}=\frac{P}{Q}=\frac{0}{0}:$$

but it may be observed, as in (233) that $\frac{dy}{dx}=\frac{0}{0}$, does not in all cases necessarily indicate the existence of a point of this description.

236. *To determine the multiple Points of a curve formed by the Contacts of its different Branches.*

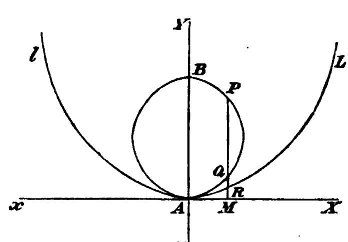
If $u=0$ be the equation of a curve proposed, it is obvious from what has already been said, that the co-ordinates of a multiple point of this second species must likewise satisfy simultaneously the three equations

$$u=0, \quad \frac{du}{dx}=0 \text{ and } \frac{du}{dy}=0;$$

and to ascertain the nature and degree of the contact, it must be determined how many of the differential coefficients belonging to the different branches become equal to one another.

Ex. Let the curve be defined by the equation

$$x^4-ax^2y+\tfrac{1}{5}a^2y^2-ay^3=0:$$



then at a multiple point of contact we must have

$$\frac{du}{dx} = 4x^3 - 2ayx = 0 \text{ and } \frac{du}{dy} = -ax^2 + \tfrac{2}{5}a^2y - 3ay^2 = 0,$$

which together with the proposed equation $u = 0$, will both be satisfied by the co-ordinates $x = 0$ and $y = 0$: and therefore the origin may be a multiple point: but we have here

$$\frac{dy}{dx} = -\frac{4x^3 - 2ayx}{ax^2 - \frac{2}{5}a^2y + 3ay^2} = \frac{0}{0},$$

from which, by the ordinary process, we find $\frac{dy}{dx} = 0$: and this leads us to conclude that the curve has but one tangent at the origin which is the axis of x: and proceeding in the same manner, we should discover that the second differential coefficient $\frac{d^2y}{dx^2} = \frac{0}{0}$ admits of two different values, so that the origin is a double point at which the two branches of the curve have merely simple contact, as in the figure annexed.

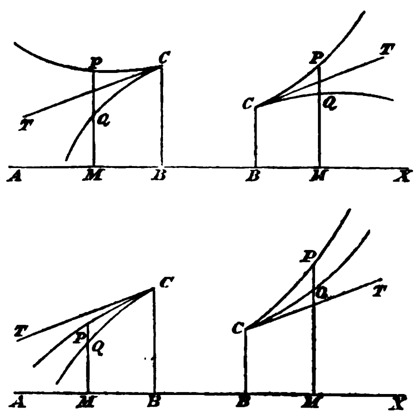
A point of this description is, by English Mathematicians, generally termed a *Point of Osculation* whose order is the same as that of the contact subsisting between the branches, and in the works of the French writers it is styled *Un Point d'Embrassement.*

237. It scarcely need be observed here, as it must have already appeared from the diagrams in the preceding pages that a double point is most commonly accompanied by an oval termed a *Single Node:* and in the same manner a triple point has frequently a *Double Node* attendant upon it, and so on.

(3) *Points of Reflexion commonly called Cusps.*

238. If when a certain value is assigned to the principal variable, it appears that y and $\frac{dy}{dx}$ have each but one value, and one of the quantities $f(x+h)$ or $f(x-h)$ becomes imaginary, however small h may be assumed, whilst $\frac{d^2y}{dx^2}$ has two

different values, the corresponding point in the curve is a point of reflexion, and is styled a cusp of the first or second species according as the said two values of $\frac{d^2y}{dx^2}$ have different or the same algebraical signs. These circumstances are exhibited at the points C in the following diagrams.



These points being formed by the union of two branches of the curve which have a common tangent, will obviously possess the characters of multiple points of the second species discussed under the preceding head, and their existence must therefore be determined by means of the three equations

$$u=0, \quad \frac{du}{dx}=0 \text{ and } \frac{du}{dy}=0,$$

subject to the further conditions that the corresponding value of either $f(x+h)$ or $f(x-h)$ must become imaginary, and $\frac{d^2y}{dx^2}$ then have two different values. This will best appear by Examples.

Ex. 1. In the Cissoid of *Diocles* we have $y^2(2a-x)=x^3$; and thence we find the characterising equations to be

$$u = 2ay^2 - xy^2 - x^3 = 0,$$

$$\frac{du}{dx} = -y^2 - 3x^2 = 0,$$

$$\frac{du}{dy} = 4ay - 2xy = 0:$$

and it is readily seen that $x=0$ and $y=0$ satisfy them all:

now $\frac{dy}{dx} = \frac{(3a-x)\,x^{\frac{1}{2}}}{(2a-x)^{\frac{3}{2}}} = 0$ at this point, and therefore the curve has here but one tangent which is the axis of x:

also, if $0-h$ be put for x, the ordinate becomes imaginary, and for $x=0+h$, we find that $\frac{d^2y}{dx^2}$ possesses the two unequal values

$$\frac{3a^2}{h^{\frac{1}{2}}(2a-h)^{\frac{5}{2}}} \text{ and } -\frac{3a^2}{h^{\frac{1}{2}}(2a-h)^{\frac{5}{2}}}:$$

and these circumstances indicate the existence of a cusp of the first species at the origin of the co-ordinates, as is well known to be the case.

Ex. 2. The equation to the common Cycloid is

$$y = a \text{ vers}^{-1}\frac{x}{a} + \sqrt{2ax - x^2}:$$

whence $\frac{dy}{dx} = \sqrt{\frac{2a-x}{x}}$ and $\frac{d^2y}{dx^2} = -\frac{a}{x\sqrt{2ax-x^2}}$:

now by means of the characteristic equations it is easily shewn that $x=2a$ and $y=\pi a$ are the only pair of co-ordinates which can belong to a multiple point: and these give $\frac{dy}{dx}=0$, or the curve has only one tangent there which is parallel to the axis

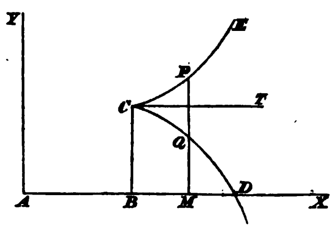
of x: also, for $x=2a+h$ the value of y is impossible, and the values of $\frac{d^2y}{dx^2}$ corresponding to $x=2a-h$ are

$$\pm \frac{a}{(2a-h)\sqrt{2ah-h^2}},$$

so that the point where the curve meets its extreme ordinate is a cusp of the first species.

Ex. 3. Let the equation of the curve be $(y-b)^2=(x-a)^3$, whence we find

$$y=b\pm(x-a)^{\frac{3}{2}},\ \frac{dy}{dx}=\pm\frac{3}{2}(x-a)^{\frac{1}{2}} \text{ and } \frac{d^2y}{dx^2}=\pm\frac{3}{4}(x-a)^{-\frac{1}{2}}:$$

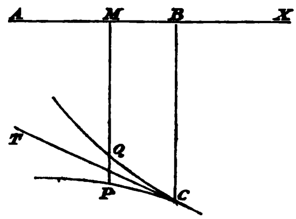


now if $x=a$, we have $y=b$ and $\frac{dy}{dx}=0$: that is, at this point there is only one value of the ordinate and one tangent which is parallel to the axis of x: also, for $x=a-h$, the ordinates are imaginary, and the values of $\frac{d^2y}{dx^2}$ become $=\pm\frac{3}{4\sqrt{h}}$ when the abscissa is $a+h$. These circumstances determine the existence of a cusp at C, as in the annexed diagram, where the upper branch CE which is convex towards the axis of x is pointed out by $\frac{3}{4\sqrt{h}}$ and the lower CD which is concave by $-\frac{3}{4\sqrt{h}}$.

Ex. 4. If $(2y+x+a)^2=2(a-x)^5$, we shall have

$$\frac{dy}{dx}=-\frac{2y+x+a+5(a-x)^4}{2(2y+x+a)}:$$

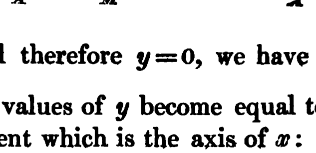
now if $x=a$, we find that y has but one value $-a$, and $\frac{dy}{dx}$



then becomes $=-\frac{1}{2}$: also for $x=a+h$ the values of y are impossible, and $\frac{d^2y}{dx^2}$ has two unequal values for $x=a-h$: consequently if $AB=a$ and $BC=a$, the point C will be a cusp of the first species, the rectilineal tangent CT common to the two branches being inclined to the axis of x at an angle whose trigonometrical tangent $=-\frac{1}{2}$.

Ex. 5. Let the equation be $(y-ax^2)^2=bx^5$, or $y=ax^2 \pm bx^{\frac{5}{2}}$: then

$$\frac{dy}{dx}=2ax\pm\tfrac{5}{2}bx^{\frac{3}{2}} \text{ and } \frac{d^2y}{dx^2}=2a\pm\tfrac{15}{4}bx^{\frac{1}{2}};$$

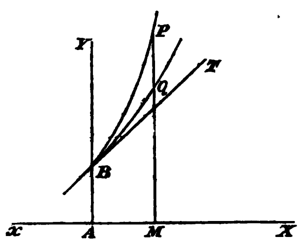


if now $x=0$ and therefore $y=0$, we have $\frac{dy}{dx}=0$, or at this point where two values of y become equal to each other, there is only one tangent which is the axis of x: also for $x=0-h$, the values of y are impossible, and corresponding to $x=0+h$, we find the values of y to be $2a+\frac{15}{4}bh^{\frac{1}{2}}$ and $2a-\frac{15}{4}bh^{\frac{1}{2}}$,

which are both positive when h is a very small magnitude: whence at the origin there is obviously an union of two branches of the curve both convex towards the axis of x forming what is called a cusp of the second species, as exhibited in the diagram.

Ex. 6. Let $y = a + x + bx^2 + cx^{\frac{5}{2}}$: then we have

$$\frac{dy}{dx} = 1 + 2bx + \frac{5}{2}cx^{\frac{3}{2}} \text{ and } \frac{d^2y}{dx^2} = 2b + \frac{15}{4}cx^{\frac{1}{2}}:$$



and supposing $x = 0$, and therefore $y = a$, we obtain $\frac{dy}{dx} = 1 = \tan 45^0$: that is at this point, the two values of the ordinate merge into one, and the branches of the curve have there a common rectilineal tangent inclined at an angle of 45^0 to the axis of x: also, when $x = 0 - h$, the values of y are impossible, and when $x = 0 + h$, $\frac{d^2y}{dx^2}$ has the two unequal values

$$2b + \frac{15c}{4}\sqrt{h} \text{ and } 2b - \frac{15c}{4}\sqrt{h},$$

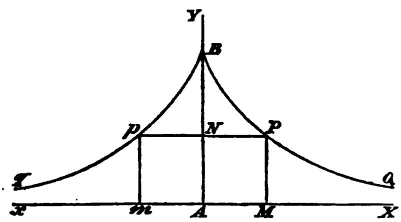
which are both manifestly positive when h is very small. This point is therefore a cusp of the second species, as delineated in the figure, both of the branches being convex towards the axis of x.

239. To ascertain the natures and positions of all the cusps that may belong to any curve whose equation is given, it is obvious that the process above used with reference to the axis of x, must be repeated with respect to the axis of

y where $x=f(y)$, and the differential coefficients are $\dfrac{dx}{dy}$ and $\dfrac{d^2x}{dy^2}$, as in the following instance.

Ex. For the Tractrix in Ex. 8 of (177) whose equation is

$$\frac{x+\sqrt{a^2-y^2}}{a}=\log\left(\frac{a+\sqrt{a^2-y^2}}{y}\right),$$



we have

$$\frac{dx}{dy}=-\frac{\sqrt{a^2-y^2}}{y} \text{ and } \frac{d^2x}{dy^2}=\frac{a^2}{y^2\sqrt{a^2-y^2}}:$$

and if we make $y=a$, and therefore $x=0$, we find $\dfrac{dx}{dy}=0$, so that at this point the two values of both x and $\dfrac{dx}{dy}$ coincide in one: this is therefore a multiple point to which belongs only one tangent coincident with the axis of y:

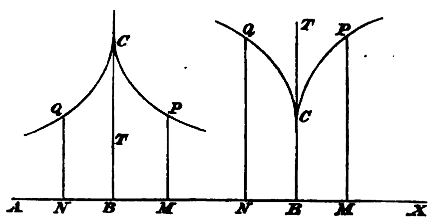
also, if for y we put $a+h$, the values of x are impossible, and corresponding to the value $a-h$ of y, the differential coefficient $\dfrac{d^2x}{dy^2}$ has the two different values

$$\frac{a^2}{(a-h)^2\sqrt{2ah-h^2}} \text{ and } -\frac{a^2}{(a-h)^2\sqrt{2ah-h^2}}$$

when h is very small, which prove the cusp to be of the first species, as at the point B in the annexed diagram.

240. Cor. 1. In cases similar to the example just given wherein $\dfrac{dx}{dy}=0$, we shall obviously have $\dfrac{dy}{dx}=\infty$, $\dfrac{d^2y}{dx^2}=\infty$, &c.

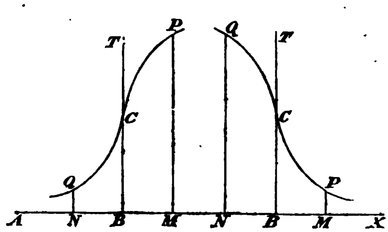
and though we have here been assured of the existence of a cusp by referring the curve to the axis of y, it is manifest that this point does not answer in its conditions to the analytical character of a cusp with respect to the axis of x, as is evident in either of the following figures, wherein neither



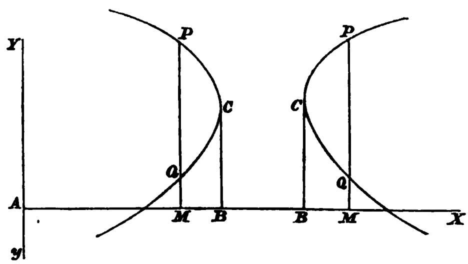
$f(x+h)$ nor $f(x-h)$ becomes impossible, nor does $\dfrac{d^2y}{dx^2}$ take two different values corresponding to one value of x.

Whenever therefore we have $\dfrac{dy}{dx} = \infty$ and $\dfrac{d^2y}{dx^2} = \infty$, and it appears that a root of the equation $\dfrac{dy}{dx} = \infty$ renders y a maximum or a minimum, the corresponding point in the curve will be a cusp of the first species as exhibited in the first or second parts of the figure, according as $f(x+h)$ and $f(x-h)$ are then both less or both greater than $f(x)$.

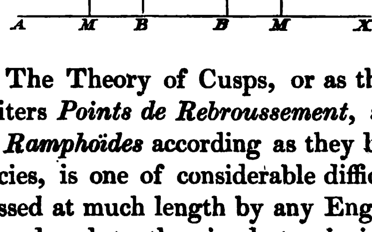
241. Cor. 2. If this value of the second differential coefficient $\dfrac{d^2y}{dx^2}$ become positive for one of the values $y=f(x+h)$ and $y=f(x-h)$ and negative for the other, it is evident from (220) that the corresponding point will be one of contrary flexure and not a cusp, as in the following diagrams: but



if either of the said quantities become imaginary on the same hypothesis, we shall manifestly have a limit of the curve in the direction of the axis of x, as in the figure underneath:



which belongs to a maximum or minimum of x, and the corresponding ordinate is a maximum or a minimum only in the sense which has been briefly explained and illustrated in (124) and (125): or there may exist a cusp of the second species determinable by the ordinary process, as in this diagram.



242. The Theory of Cusps, or as they are called by the French writers *Points de Rebroussement*, and sometimes *Ceratoïdes* and *Ramphoïdes* according as they belong to the first or second species, is one of considerable difficulty, and has never been discussed at much length by any English author. It has here been reduced to the simplest principles that the subject apparently admits of, and their application will, it is presumed, enable the student to determine the positions of such points of a curve whenever they exist: and the following proposition will place in a still clearer light some circumstances which may have been previously but slightly touched upon.

243. ***To determine the nature of any proposed Point of a curve whose equation is given.***

Transferring the origin of the co-ordinates to the point under consideration, let us suppose the resulting equation to be reduced to the form

$$y = Ax^{\alpha} + Bx^{\beta} + Cx^{\gamma} + \&c.$$

wherein α, β, γ, &c. are increasing positive quantities: then we have

$$p = \alpha Ax^{\alpha-1} + \beta Bx^{\beta-1} + \gamma Cx^{\gamma-1} + \&c.$$

$$q = \alpha(\alpha-1)Ax^{\alpha-2} + \beta(\beta-1)Bx^{\beta-2} + \gamma(\gamma-1)Cx^{\gamma-2} + \&c.$$

&c.......................

whence diminishing indefinitely the corresponding values of x and y, we shall be enabled to ascertain the nature of the point proposed by an examination of these expressions: and with this view we shall consider in order the effects of the successive indices α, β, γ, &c.

244. To estimate the effect of the first index α, we have

$$y = Ax^{\alpha},$$

$$p = \alpha Ax^{\alpha-1},$$

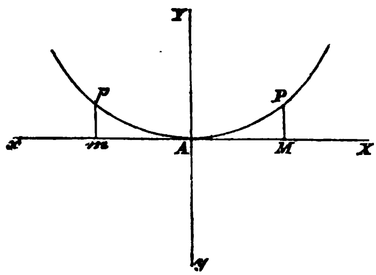
$$q = \alpha(\alpha-1)Ax^{\alpha-2}:$$

and α may be greater than, equal to, or less than 1.

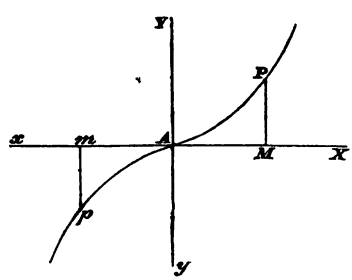
When α is greater than 1.

First, let α be of the form 2μ or $\frac{2\mu}{2\nu+1}$, μ and ν being whole numbers: then it is obvious that y will be positive whether x be a positive or negative magnitude: also, when $x = 0$,

we have $p=0$, so that the axis of x is a tangent to the curve as at the proposed point A, and the ordinate is there a minimum.

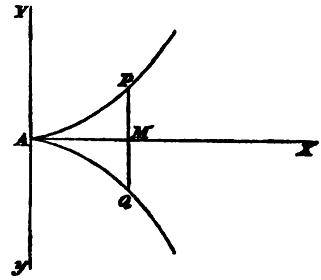


Secondly, let a be of the form $2\mu+1$ or $\frac{2\mu+1}{2\nu+1}$; then it is manifest that y will be positive or negative according as x is positive or negative; and since when $x=0$, $p=0$, the axis of x will be a tangent to the curve passing through a point of inflexion, because y and q have always the same algebraical sign.



Thirdly, let a be of the form $\frac{2\mu+1}{2\nu}$, then will y have two values, one positive and one negative when x is positive, but no possible values when x is negative: from the proposed point there will therefore issue two branches of the curve on the same side of the axis of y, touching on opposite sides the axis of x since $p=0$, and both of the branches will be

convex to it because the corresponding values of y and q have still the same algebraical sign.



When a is equal to 1.

In this case we have $y = Ax$ and $p = A$, so that the curve here crosses the axis of x at an angle whose trigonometrical tangent is A, and no inflexion is indicated there since $q = 0$ whether x and y be positive or negative.

When a is less than 1.

Here from the equation $y = Ax^a$, we have immediately $x = \dfrac{1}{A^{\frac{1}{a}}} y^{\frac{1}{a}}$, where $\dfrac{1}{a}$ is obviously greater than 1: and the discussion of this case with reference to the axis of y in the same manner as has just been done with respect to that of x will determine the nature of the proposed point.

245. To consider the effect of the second index β, we have

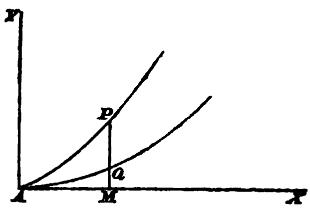
$$y = Ax^{\alpha} + Bx^{\beta},$$

$$p = \alpha A x^{\alpha-1} + \beta B x^{\beta-1},$$

$$q = \alpha(\alpha-1) A x^{\alpha-2} + \beta(\beta-1) B x^{\beta-2}.$$

First, let α be greater than 1, and let β be of the form $\dfrac{2\mu+1}{2\nu}$: then it is manifest that when x is negative, y becomes

imaginary, so that the curve does not cross the axis of y: also because $p=0$ when $x=0$, and y has two values corresponding to one value of x and q is positive for both, the point of the curve is a cusp of the second species, similar to A in the following diagram.



Secondly, let $a=1$ and β be of the form $\dfrac{2\mu+1}{2\nu}$; then will y become imaginary when x is negative: also when $x=0$, $p=0$, and of the two values of q corresponding to one of x, one will be positive and the other negative; therefore the point proposed is a cusp of the first species, as at A of the figure in the last page.

If β were assumed to be of any of the forms 2μ, $\dfrac{2\mu}{2\nu+1}$, $2\mu+1$, $\dfrac{2\mu+1}{2\nu+1}$, no peculiarity of figure would present itself in addition to what is indicated in the different cases of $y=Ax^{a}$ above considered.

246. To consider the effect of the third index γ, we have

$$y=Ax^{\alpha}+Bx^{\beta}+Cx^{\gamma},$$
$$p=aAx^{\alpha-1}+\beta Bx^{\beta-1}+\gamma Cx^{\gamma-1},$$
$$q=a(a-1)Ax^{a-2}+\beta(\beta-1)Bx^{\beta-2}+\gamma(\gamma-1)Cx^{\gamma-2};$$

and if γ be of the form $\dfrac{2\mu+1}{2\nu}$, the curve does not cross the axis of y, but y has two values corresponding to one of x,

so that there will be a cusp of the first or second species, according as q has different or the same algebraical signs for both.

If γ be of any of the forms 2μ, $\frac{2\mu}{2\nu+1}$, $2\mu+1$, $\frac{2\mu+1}{2\nu+1}$, no additional singularity occurs.

In this case, since the two branches of the curve are supposed to originate from the third term of the developement, it is obvious that they have with each other contact of the second order, because the first and second differential coefficients will be the same for both when x is diminished without limit, and the curve whose equation is $y=Ax^{\alpha}+Bx^{\beta}$ will manifestly pass between the arcs which form the cusp.

247. Cor. We might proceed in the same manner to estimate the effects of still higher indices, and should the developement of y comprise m exponents of the form $\frac{2\mu+1}{2\nu}$ independent of one another, it will obviously follow that y has several different values arising from the different combinations of the expressions that originate by means of the double signs of the radical quantities.

248. In these latter articles we have considered all the coefficients A, B, C, &c. to be positive quantities: but should any of them become negative, the principles explained will still be quite sufficient, the only alteration necessary being a change in the direction of the branches issuing from the point proposed.

Should however one or more of these quantities become imaginary, their effect may be to render real some of the terms which we have regarded as imaginary when negative values have been assigned to x: but the result being merely a change in the algebraical sign of such term will form no exception to what has been already said: and when this effect does not take place, we have been informed in (226) that the point under discussion is a conjugate point.

249. If a curve possess a point distinguished by any singularity, it is easily shewn that some kind of singularity will be attendant upon the corresponding point of its evolute.

Thus, if α, β be the co-ordinates of the evolute corresponding to x, y of the proposed curve, we have seen that

$$\alpha = x - \frac{p(1+p^2)}{q} \text{ and } \beta = y + \frac{1+p^2}{q}:$$

but at a point of contrary flexure it has been proved that $q = 0$ or ∞ : therefore in the former case we have

$$\alpha = x - \infty \text{ and } \beta = y + \infty,$$

which shew that the evolute has two infinite arcs: and in the latter we find

$$\alpha = x \text{ and } \beta = y,$$

so that the curve and its evolute have a point in common, and since the direction of curvature changes at a point of inflexion, it follows that the evolute has also in this case a point of contrary flexure.

At a cusp of the first species where the radius of curvature undergoes a change in its algebraical sign, it is obvious that $\gamma = 0$ or ∞, and therefore the evolute will possess either two infinite arcs or a cusp of the same kind, as may readily be seen by constructing the figure.

At a cusp of the second species however, the curvature undergoes no such change, and it may without much difficulty be proved that the corresponding point in the evolute is an inflexion, the tangent at which is a normal to the cusp.

250. If the curve proposed be defined by polar co-ordinates, the relation between its rectangular co-ordinates may with great ease be ascertained, and the singular points with which the curve is attended may then be investigated according to the principles we have just developed. Without however recurring to such transformation, it is no difficult matter to

determine several of them: such as points corresponding to a maximum or minimum radius vector and points of inflexion, the former of which may manifestly be found by assuming $\frac{dr}{d\theta}=0$, which will give a maximum or minimum value of r according as $\frac{d^2r}{d\theta^2}$ is then negative or positive, and the latter will be determined in the following article.

251. *To find whether a polar curve has a point of Inflexion, and to determine its position.*

Retaining the notation hitherto used in polar curves, we have seen in (201) that if a spiral be concave towards its pole or radius vector, the value of $\frac{dp}{dr}$ is positive, and if it be convex, negative: hence it follows that at a point of contrary flexure, where the curve passes from concave to convex, or from convex to concave, $\frac{dp}{dr}$ must change its algebraical sign, and consequently at such a point pass through zero or infinity. Hence therefore to determine the points of inflexion in a polar curve, we must put $\frac{dp}{dr}=0$ or ∞, and the real roots of these equations which satisfy also the equation of the curve, will be the values of the radii vectores corresponding to them.

Ex. 1. In the Lituus we have $r=a\theta^{-\frac{1}{2}}$;

$$\therefore\ p=\frac{2a^2r}{\sqrt{4a^4+r^4}} \text{ and } \frac{dp}{dr}=\frac{2a^2(4a^4-r^4)}{(4a^4+r^4)^{\frac{3}{2}}}:$$

now if $4a^4$ be greater than r^4, or r be less than $a\sqrt{2}$, this spiral is concave towards its pole:

if $\frac{dp}{dr}=0$ and therefore $r=a\sqrt{2}$, there is a point of inflexion:

and if $4a^4$ be less than r^4, or r be greater than $a\sqrt{2}$, the spiral is convex towards the radius vector.

The characteristic equations might have been satisfied in other ways, but the corresponding values of r would have been either infinite or impossible.

Ex. 2. Let the equation to the spiral be $r = \dfrac{a\theta^2}{\theta^2 - 1}$;

$$\text{then } p = \frac{ar^2}{\sqrt{a^2r^2 + 4r\,(r-a)^3}},$$

$$\text{and } \frac{dp}{dr} = \frac{a^2 r^{\frac{1}{2}}\,(6r^2 - 13ar + 6a^2)}{\{a^2 r + 4\,(r-a)^3\}^{\frac{3}{2}}};$$

now $\dfrac{dp}{dr}$ being put $=0$, gives

$$6r^2 - 13ar + 6a^2 = 0$$

whose roots are $\frac{3}{2}a$ and $\frac{2}{3}a$:

but since the latter value of r would render θ impossible, the only point of contrary flexure in this spiral is at the distance $\frac{3}{2}a$ from the pole.

CHAP. XI.

On the Application of the Differential Calculus to the Describing or Tracing of Plane Curves.

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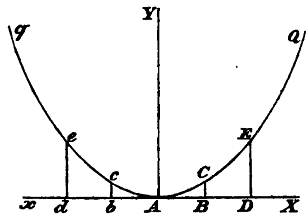
252. WHEN a curve is defined by an equation of the form $y=f(x)$ or $f(x, y)=0$, the theory of equations informs us, that whatever value is assigned to x, there will be as many correspondent values of y either real or imaginary, as the equation, when reduced to the ordinary form, has dimensions. By giving to x therefore all possible degrees of magnitude, both positive and negative, we shall be enabled to find as many points in the curve proposed as we please, and by joining the points belonging to every contiguous pair of real co-ordinates thus found, to *describe* or *trace* it.

The result however of this kind of construction will obviously not be a curve, but a polygonal figure which approaches more and more nearly to that of the curve proposed, the more the number of the co-ordinates taken is increased; and it will readily be observable that the want of continuity in the arcs of curves thus described by means of points, will render it impossible for the ordinary principles of either Algebra or Geometry to ascertain with much accuracy the natures and circumstances of any proposed parts of them. To the determination of such circumstances the *Differential Calculus* has already been directed in some of the preceding Chapters, and the present Chapter will consist merely of Examples of Curves, in which the principles before laid down, are, as it were, brought together and applied.

Ex. 1. Trace the curve whose equation is $ay=x^2$.

Let AX, AY be the axes of x and y respectively and A the origin of co-ordinates: then if $x=0$, we have $y=0$ or



the curve passes through A: if $x=1$, then $y=\frac{1}{a}$; if $x=2$, $y=\frac{4}{a}$ &c.: therefore if AB, AD, &c. be taken equal to 1, 2, &c. respectively, and the corresponding ordinates BC, DE, &c. be made equal to $\frac{1}{a}$, $\frac{4}{a}$, &c. the curve will pass through the points C, E, &c.: also, as x increases in infinitum, y is positive and increases in infinitum, or the curve is indefinitely extended in its branch $ACEQ$:

next let x be negative and taken equal to -1, -2, &c. in succession, and the value of y becomes equal to $\frac{1}{a}$, $\frac{4}{a}$, &c. in order: whence if Ab, Ad, &c. be made equal to 1, 2, &c. and the ordinates bc, de, &c. be drawn, the curve passes through c, e, &c., and whilst x is negative and increases indefinitely, y is positive and increases indefinitely, so that the curve is infinite in its branch $Aceq$:

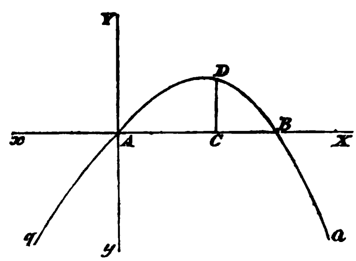
again, since $ay=x^2$, we have $\frac{dy}{dx}=\frac{2x}{a}$: and therefore at A, $\tan X=0$ or the axis of x touches the curve at the origin of the co-ordinates: and from $\frac{dy}{dx}=0$, we find $x=0$ corresponding to a minimum ordinate at A:

also, since $\frac{d^2 y}{dx^2} = \frac{2}{a}$ is a positive quantity, and the ordinate y is always positive, it follows that the curve is always convex towards the axis of x, and does not admit of a point of contrary flexure.

The equation to the tangent $y' - y = p\,(x' - x)$ here becomes $y' = \frac{2x}{a}x' - \frac{x^2}{a}$, from which the rectilineal tangent at any proposed point may be constructed, and the curve cannot have an asymptote since when x is infinite, no finite relation subsists between x' and y'.

Ex. 2. Let the equation to the curve be $a^3 y = x\,(b^3 - x^3)$.

Take the axes and origin as before; then if $x = 0$, $y = 0$,



or the curve passes through A: whilst x is positive and less than b, y is positive, and the curve corresponding lies above the axis xAX, which it meets at B if AB be taken $= b$, since $y = 0$ when $x = b$: when x becomes greater than b and increases in infinitum, y becomes negative and increases indefinitely, or the branch BQ of the curve is infinite:

when x becomes negative and increases indefinitely, y becomes negative and does so likewise, so that the branch Aq lying below the axis of x is also indefinitely extended:

again, since $\frac{dy}{dx} = \frac{b^3 - 4x^3}{a^3}$, at A we shall have $\tan X = \frac{b^3}{a^3}$, and at B $\tan X = -\frac{3b^3}{a^3}$ or the angles at which the curve inter-

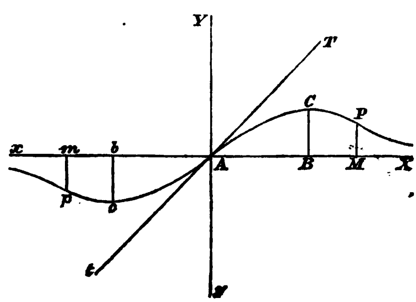
sects the axis of x are found: and from $\frac{dy}{dx}=0$, is obtained $x=\frac{b}{\sqrt[3]{4}}$; whence if $AC=\frac{b}{\sqrt[3]{4}}$ and CD be drawn, it is a maximum ordinate: also, $\frac{d^2y}{dx^2}=-\frac{12x^2}{a^3}$, which is always negative whatever be the value of x, and therefore the curve has its concavity always downwards, and admits not of a point of inflexion.

The subtangent on the axis of x expressed generally by $\frac{ydx}{dy}$ here becomes $\frac{x(b^3-x^3)}{b^3-4x^3}$, by means of which the rectilineal tangent to any proposed point of the curve may be drawn, and it can never become an asymptote since $\frac{dy}{dx}$ and $\frac{ydx}{dy}$ are infinite when x is infinite.

Ex. 3. Let the curve be defined by the equation

$$y=\frac{x}{1+x^2}.$$

Assuming the co-ordinate axes as before, when $x=0$, we



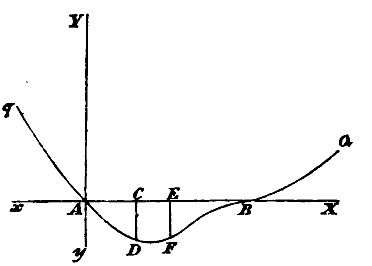
have $y=0$, and therefore the curve passes through A: when x is less than 1, y is positive or the curve lies above AX: when $x=1$, we have $y=\frac{1}{2}$, so that if $AB=1$ and $BC=\frac{1}{2}$, the curve passes through C; when x is infinitely great, y is positive

and indefinitely small, or AX becomes an asymptote to the curve in that direction: when x is negative and less than 1, y is negative and the curve lies below Ax: when $x = -1$, $y = -\frac{1}{2}$; therefore if we make $Ab = 1$ and $bc = \frac{1}{2}$, the curve passes through the point c, and when x is negative and indefinitely great, y is of an indefinitely small negative magnitude, or Ax becomes an asymptote to the curve:

also, since $\dfrac{dy}{dx} = \dfrac{1-x^2}{(1+x^2)^2}$, we have at the origin the angle $TAX = 45^0$: and $\dfrac{dy}{dx} = \dfrac{1-x^2}{(1+x^2)^2} = 0$, gives $x = \pm 1$, so that C and c are points at which the ordinates BC, bc are maxima:

again, $\dfrac{d^2y}{dx^2} = -\dfrac{2x(3-x^2)}{(1+x^2)^3}$, which will manifestly have its algebraical sign different from, or the same as that of y according as x is less than or greater than $\sqrt{3}$: hence if $AM = Am = \sqrt{3}$, and the ordinates MP, mp be drawn, the arcs ACP, Acp will be concave towards the axis of x, and P, p being points of contrary flexure, the branches of the curve there become convex towards the same axis.

Ex. 4. Let $a^3y = x(x-b)^3$; then when $x = 0$, we have $y = 0$, or the curve passes through A: when x is less than b,



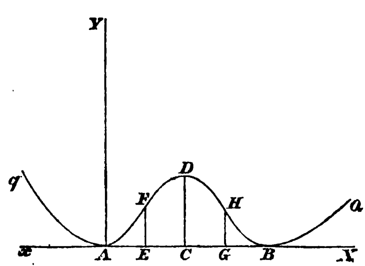
y is negative and the curve lies below AX: when $x = b$, $y = 0$: take therefore $AB = b$ and the curve passes through B: when x is greater than b and becomes infinite, y is positive and becomes infinite, or the arc BQ is infinite in extent: whilst x

is negative and becomes infinite, y is positive and becomes infinite or the curve is infinite in its branch Aq:

again, $\frac{dy}{dx} = \frac{(x-b)^2(4x-b)}{a^3}$: therefore at A where $x=0$, we have $\tan X = -\frac{b^3}{a^3}$; and at B where $x=b$, we find $\tan X=0$, so that the axis of x is a tangent to the curve there; from $\frac{dy}{dx}=0$, we have $x=b$ and $x=\frac{1}{4}b$: whence if $AC=\frac{1}{4}b$ and the ordinate CD be drawn, it will be a maximum in the direction of Ay, and $x=b$ belongs neither to a maximum nor a minimum, as $\frac{d^2y}{dx^2}$ will readily shew:

also, $\frac{d^2y}{dx^2} = \frac{6(x-b)(2x-b)}{a^3}$ and $\frac{d^3y}{dx^3} = \frac{24x-18b}{a^3}$: therefore when $x=\frac{1}{2}b$ or $x=b$, we have $\frac{d^2y}{dx^2}=0$ and $\frac{d^3y}{dx^3}$ remains finite, so that if $AE=\frac{1}{2}b$ and EF be drawn $=\frac{1}{16}\frac{b^4}{a^3}$, F and B are points of contrary flexure, the direction of the curvature of the curve changing at each of these points, and the convexity or concavity of the arc depending upon the sign of $\frac{d^2y}{dx^2}$, as in (155).

Ex. 5. Let $a^3y=x^2(x-b)^2$: then when $x=0$, $y=0$, or the curve passes through A: whilst x is less than b, y is positive



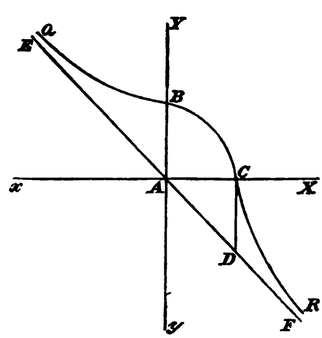
and the curve lies above the axis of x: when $x=b=AB$, the curve passes through B, and when x is positive and infinite, y is so too, or the curve is indefinitely extended towards Q: when x is negative and becomes infinite, y is positive and infinite, therefore the curve is infinite also towards Aq:

again, $\dfrac{dy}{dx}=\dfrac{2x(x-b)(2x-b)}{a^3}$, which at the points A and B gives $\tan X=0$, and therefore the axis of x is at those points a tangent to the curve: from $\dfrac{dy}{dx}=0$ we have $x=0$, $x=b$ and $x=\frac{1}{2}b$, the first two of which are easily shewn to give minima values of the ordinate, and the last points out a maximum whose co-ordinates are $AC=\frac{1}{2}b$ and $CD=\dfrac{b^4}{16a^3}$:

also, $\dfrac{d^2y}{dx^2}=\dfrac{2(6x^2-6bx+b^2)}{a^3}$ and $\dfrac{d^3y}{dx^3}=\dfrac{12(2x-b)}{a^3}$: therefore from $\dfrac{d^2y}{dx^2}=0$, we have $x=\dfrac{a(\sqrt{3}\pm1)}{2\sqrt{3}}$, corresponding to which as at F and H are points of contrary flexure, the arcs qAF, HBQ being convex and the arc FDH concave towards the axis of x.

A tangent may easily be drawn to any point of the curve, and it does not admit of an asymptote.

Ex. 6. Let $y^3=a^3-x^3$: then when $x=0$, $y=a$: take



therefore $AB=a$ and the curve passes through B: when $x=a$, $y=0$; therefore take $AC=a$, and the curve passes through C: when x is $>a$ and becomes infinite, y is $-$ and becomes infinite; therefore the arc CR is below xAX and indefinite in extent: when x becomes $-$, y is $+$, and they become infinite together, therefore the branch BQ is indefinitely extended:

also, $\dfrac{dy}{dx}=-\dfrac{x^2}{(a^3-x^3)^{\frac{2}{3}}}$: whence at C where $x=a$, we have $\tan X=\infty$ or $X=90^0$, and at C the curve cuts the axis of x at right angles: and $\dfrac{dy}{dx}=0$ gives $x=0$, which causing $\dfrac{d^2y}{dx^2}$ to become $=0$, does not belong to a maximum or minimum of y:

again, $\dfrac{d^2y}{dx^2}=-\dfrac{2a^3x}{(a^3-x^3)^{\frac{5}{3}}}$ and $\dfrac{d^3y}{dx^3}=-\dfrac{2a^3(4x^3-a^3)}{(a^3-x^3)^{\frac{8}{3}}}$:

but when x is $-$, $\dfrac{d^2y}{dx^2}$ is $+$,

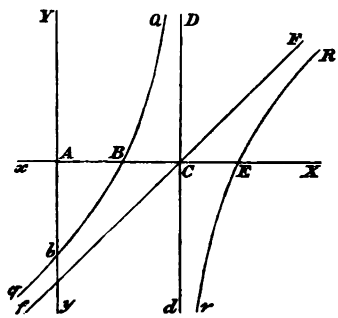
when x is $+$ and $<a$, $\dfrac{d^2y}{dx^2}$ is $-$,

when x is $+$ and $>a$, $\dfrac{d^2y}{dx^2}$ is $+$;

therefore the arc QB is convex, BC concave and CR concave towards xAX: and when $x=0$ or a, we have $\dfrac{d^2y}{dx^2}=0$ or ∞, and therefore at B and C are points of inflexion:

also, since $y=-x\left(1-\dfrac{a^3}{3x^3}+\&c.\right)$, the equation to the rectilineal asymptote will be $y'=-x'$, which may manifestly be constructed by making $CD=AC$, joining AD and producing the line indefinitely both ways.

Ex. 7. Let $y = \dfrac{(x-1)(x-3)}{x-2}$; then when $x=0$, $y=-\frac{3}{2}$:



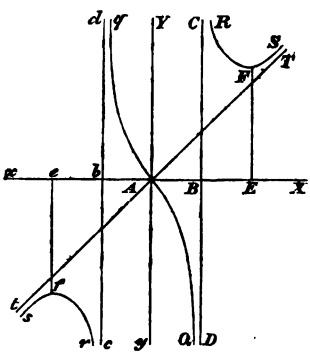
take therefore $Ab=\frac{3}{2}$, and the curve passes through b: when x is <1, y is $-$, and therefore the curve lies below xAX: when $x=1$, $y=0$: take therefore $AB=1$, and the curve passes through B: when x is between 1 and 2, y is $+$, and therefore the curve lies above xAX: when $x=2$, $y=\infty$: take therefore $AC=2$, and the indefinite ordinate CD will be an asymptote to the curve: whilst x increases in magnitude and when $x=3$, $y=0$: from 2 to 3, y is $-$ and decreases in magnitude from ∞ to 0, therefore take $AE=3$, and the curve passes through E from below xAX: when x is >3 and becomes infinite, y is $+$ and becomes infinite, and the arc ER is infinite: when x is $-$ and becomes infinite, y is $-$ and becomes infinite, and the arc bq is infinite:

again, $\dfrac{dy}{dx} = \dfrac{x^2-4x+5}{(x-2)^2}$; therefore at B where $x=1$, we have $\tan X=2$, and at E where $x=3$, $\tan X=2$, or the branches at B and E are parallel to each other: if $\dfrac{dy}{dx}=0$, we have $x^2-4x+5=0$, which gives $x=2\pm\sqrt{-1}$, so that there is neither a maximum nor a minimum ordinate:

also, $\dfrac{d^2y}{dx^2} = -\dfrac{2}{(x-2)^3}$, which is always $+$ when x is $+$ and <2, always $-$ when x is $+$ and >2, and always $-$ when

x is $-$: therefore the branches Rr and Qq are convex towards each other: and $\frac{d^2y}{dx^2}=0$ shews that there is inflexion only at an infinite distance: also, since $y=x-2-\frac{1}{x}-\frac{2}{x^2}-\&c.$ the curve admits of a rectilineal asymptote FCf whose equation is $y'=x'-2$ by (153), and which may therefore readily be constructed.

Ex. 8. Let $y=\frac{x(x^2+1)}{x^2-1}$: then when $x=0$, $y=0$, and therefore the curve passes through A: when x is <1, y is $-$ and the curve lies below xAX: when $x=1$, y is ∞ and $-$:



draw therefore the ordinate BD which indefinitely produced is an asymptote, the branch AQ being infinite: when x is >1, y is $+$, when $x-1$ is very small, y is very great, and when x is ∞, y is $+$ and ∞, therefore DB produced is an asymptote, and the branch RFS is infinite: when x is $-$ and <1, y is $+$, and therefore the curve lies above xAX: when $x=-1$, y is ∞; take therefore $Ab=1$ and the corresponding ordinate bd is an asymptote, the branch Aq being infinite: when x is $-$ and >1 and becomes ∞, y is $-$ and ∞, therefore the branch rfs is infinite: and when $x-1$ is very small, y is very great, therefore db produced is an asymptote to this branch:

again, $\dfrac{dy}{dx} = \dfrac{x^2(x^2-4)-1}{(x^2-1)^2}$, at the point of intersection A with the axis of x, gives $\tan X = -1 = \tan 135^0$, or the curve cuts that axis at an angle of 135^0: and the equation $\dfrac{dy}{dx} = 0$ gives

$$x = \pm\sqrt{2+\sqrt{5}}:$$

whence if $AE = Ae = \sqrt{2+\sqrt{5}}$, the corresponding ordinates EF, ef will be minima as easily proved:

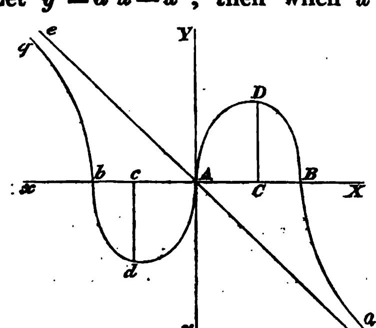
also, $\dfrac{d^2y}{dx^2} = \dfrac{2x(x^4-2x^2+5)}{(x^2-1)^2}$ which is $-$ when x is $+$ and <1 or $-$ and >1, and $+$ when x is $+$ and >1 or $-$ and <1: therefore every part of the curve is convex towards the axis of x, the upper parts having their convexity downwards and the lower upwards; but since

$$\frac{d^3y}{dx^3} = -\frac{2(x^6-x^4+19x^2+5)}{(x^2-1)^4},$$

and when $x=0$, $\dfrac{d^2y}{dx^2}=0$, but $\dfrac{d^3y}{dx^3}$ remains finite, at A there is a point of contrary flexure, which is the only one in the curve:

and since $y = x + \dfrac{2}{x} + \&c.$, the equation to the asymptote TAt is $y' = x'$, making equal angles with the axes.

Ex. 9. Let $y^3 = a^2x - x^3$; then when $x=0$, $y=0$, and



the curve passes through A: when x is $< a$, y is $+$ and the curve lies above xAX: when $x = a$, $y = 0$: therefore if $AB = a$, the curve passes through B: when x is $> a$ and ∞, y is $-$ and ∞: therefore the arc BQ lies below the axis of x and is infinite: when x is $-$ and $< a$, y is $-$ and the curve lies below xAX: when $x = -a$, $y = 0$, therefore take $Ab = a$, and the curve passes through b: when x is $-$ and $> a$ and becomes ∞, y is $+$ and becomes ∞, therefore the arc bq lies above the axis of x and is infinite:

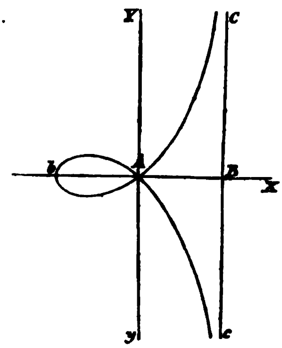
also, $\dfrac{dy}{dx} = \dfrac{a^2 - 3x^2}{3(a^2x - a^3)^{\frac{2}{3}}}$, so that at the points A, B, b, $\tan X = \infty$ and the angles of intersection are right angles: when $\dfrac{dy}{dx} = 0$, we have $x = \pm \dfrac{a}{\sqrt{3}}$ and $y = \pm a\dfrac{\sqrt[3]{2}}{\sqrt{3}}$: taking therefore $AC = Ac = \dfrac{a}{\sqrt{3}}$, and making the ordinates CD and cd each $= a\dfrac{\sqrt[3]{2}}{\sqrt{3}}$, these will be maxima in their directions; again, $\dfrac{d^2y}{dx^2} = -\dfrac{2a^2(3x^2 + a^2)}{9(a^2x - x^3)^{\frac{5}{3}}}$, which proves that the curve is always concave towards the axis of x: and when $x = 0$ or $\pm a$, we have $\dfrac{d^2y}{dx^2} = \infty$, so that the points A, B, b are all points of contrary flexure:

we find also $y = -x\left(1 - \dfrac{a^2}{3x^2} + \&c.\right)$, from which we deduce $y' = -x'$, which is the equation to the asymptote eAE equally inclined to the axes of x and y.

Ex. 10. Let $(a - x)y^2 = (a + x)x^2$, which gives

$$y = \pm x\sqrt{\frac{a + x}{a - x}}:$$

then when $x=0$, $y=0$, or the curve passes through A: when x is $+$ and $<a$, y has a positive and negative value equal to each other in magnitude, and therefore the arcs of the curve above and below the axis of x are symmetrical: when



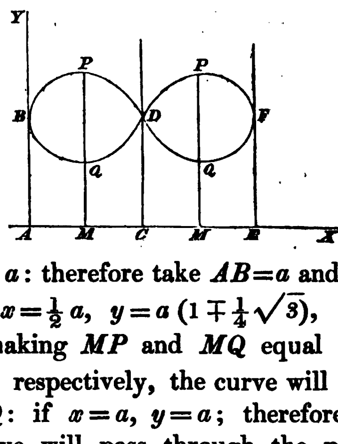
$x=a$, both the ordinates become infinite, and therefore if $AB=a$, the double ordinate CBc is an asymptote to the curve: when x is $>a$, the values of y become imaginary, and therefore no part of the curve lies to the right of CBc: when x is $-$ and $<a$, y has two equal values one $+$ and the other $-$, and the curve is symmetrical with respect to the axis of x: when $x=-a$, $y=0$, that is, if $Ab=a$, the curve passes through b: if x be $-$ and $>a$, the values of y again become imaginary, so that no part of the curve lies to the left of b:

also, $\dfrac{dy}{dx}=\pm\dfrac{a^2+ax-x^2}{a^2-x^2}\sqrt{\dfrac{a+x}{a-x}}$, which when $x=0$, gives $\tan X=\pm 1=\tan 45^0$ or $\tan 135^0$, and the tangents at the double point A are equally inclined to the co-ordinate axes: and when $x=-a$, we find that the curve at b cuts the axis at right angles: moreover $\dfrac{dy}{dx}=0$ makes $a^2+ax-x^2=0$ and $a+x=0$, from which are obtained $x=\frac{1}{2}a(1+\sqrt{5})$, $x=\frac{1}{2}a(1-\sqrt{5})$ and $x=-a$: the first of these does not belong to the curve, the second is easily shown to belong to a maximum double ordinate, and the last to neither a maximum nor a minimum:

again, $\dfrac{d^2y}{dx^2} = \pm \dfrac{a^2(2a+x)}{(a-x)(a^2-x^2)^{\frac{3}{2}}}$ has manifestly always the same sign as y, and therefore the symmetrical branches are respectively concave upwards and downwards, but $\dfrac{d^2y}{dx^2} = 0$, does not indicate a point of inflexion, nor is there any additional asymptote.

Ex. 11. Let $\dfrac{y-a}{x-a} = \pm \dfrac{\sqrt{2ax-x^2}}{a}$, from which we find

$y = a \pm \dfrac{x-a}{a}\sqrt{2ax-x^2}$;



then if $x=0$, $y=a$: therefore take $AB=a$ and the curve passes through B; if $x=\frac{1}{2}a$, $y=a(1\mp\frac{1}{4}\sqrt{3})$, therefore taking $AM=\frac{1}{2}a$ and making MP and MQ equal to $a(1+\frac{1}{4}\sqrt{3})$ and $a(1-\frac{1}{4}\sqrt{3})$ respectively, the curve will pass through the points P and Q: if $x=a$, $y=a$; therefore if $AC=a$ and $CD=a$, the curve will pass through the point D, the two ordinates there becoming equal and belonging to a double point in it, as seen in (234): if $x=\frac{3}{2}a$, $y=a(1\pm\frac{1}{4}\sqrt{3})$; whence repeating the construction for the right side of the figure, the curve will pass through P and Q; if $x=2a$, $y=a$, whence if $AE=2a$ and $EF=a$, the curve passes through F, the two ordinates here uniting in one: if x be $>2a$ or is $-$, the values of y are impossible, so that the curve does not extend to the left of AB nor to the right of EF:

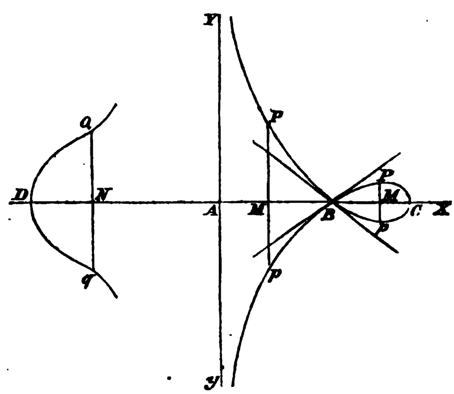
also, $\dfrac{dy}{dx} = \mp \dfrac{a^2-4ax+2x^2}{a\sqrt{2ax-x^2}}$, which at B and F, where $x=0$ and $2a$, gives $\tan X = \infty$, and at D, where $x=a$, makes

$\tan X = \pm 1 = \tan 45^{0}$ or $\tan 135^{0}$: therefore at B and F the tangents are perpendicular to the axis of x, and at the double point D the two tangents are inclined to it at 45^{0} and 135^{0}: when $\frac{dy}{dx} = 0$, we find $x = a\left(1 \pm \frac{1}{\sqrt{2}}\right)$, to each of which belong two values of y, and one of these is easily shewn to be a maximum and the other a minimum, which may readily be constructed:

again, $\frac{d^2y}{dx^2} = \mp \frac{(x-a)(a^2+4ax-2x^2)}{a^2(2ax-x^2)^{\frac{3}{2}}}$ proves that the upper part of the curve is concave towards the axis of x and the lower convex: and if $\frac{d^2y}{dx^2} = 0$, we find an inflexion of each of the branches at D from the equation $x - a = 0$, the roots of $a^2 + 4ax - 2x^2 = 0$ not belonging to the curve.

Ex. 12. Let $x^2y^2 = (a^2 - x^2)(x-b)^2$, which gives

$$y = \pm \frac{x-b}{x}\sqrt{a^2 - x^2};$$



then if $x = 0$, $y = \pm \infty$; therefore the axis of y indefinitely produced is an asymptote to the curve: if x be $+$ and $< b$, y has two values of equal magnitudes but of different algebraical signs, therefore the curve is symmetrical with respect to the axis of x: if $x = b$, $y = 0$; hence if $AB = b$, the two ordinates

vanish, and the curve passes through B, which is a double point: if x be $>b$ and $<a$, there are still two equal values of y, one + and the other −: if $x=a$, $y=0$, therefore making $AC=a$, we find the curve passes through C: if x be $>a$, the values of y become imaginary, and therefore no part of the curve lies to the right of C: if x be − and increase in magnitude, y continually decreases from ∞ having two values equal in magnitude; if $x=a$, $y=0$, therefore if $AD=a$ the curve passes through D: and if x be $>a$, y becomes imaginary, so that no part of the curve lies to the left of D:

also, $\dfrac{dy}{dx}=\pm\dfrac{a^2b-x^3}{x^2\sqrt{a^2-x^2}}$ gives at the point B where $x=b$,

$$\tan X=\pm\frac{\sqrt{a^2-b^2}}{b};$$

at C where $x=a$, $\tan X=\infty$, and at D where $x=-a$, $\tan X=\infty$: whence the curve twice intersects the axis of x in B at angles whose tangents are $\dfrac{\sqrt{a^2-b^2}}{b}$ and $-\dfrac{\sqrt{a^2-b^2}}{b}$, and at right angles in C and D: and if $\dfrac{dy}{dx}=0$, we have $x=a^{\frac{2}{3}}b^{\frac{1}{3}}$ corresponding to a maximum double ordinate:

again, $\dfrac{d^2y}{dx^2}=\mp\dfrac{a^2(2a^2b-3bx^2+x^3)}{x^3(a^2-x^2)^{\frac{3}{2}}}$ points out the convex and concave parts of the curve, and $\dfrac{d^2y}{dx^2}=0$, gives

$$x^3-3bx^2+2a^2b=0,$$

to determine the points of inflexion: and in this case there will be only two such points both belonging to the same abscissa as at Q and q.

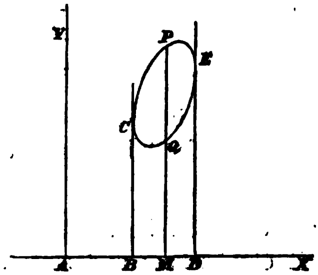
If $a=b$, or the equation be $x^2y^2=(x-b)^2(b^2-x^2)$, the node between B and C vanishes; and since at B we have then $\dfrac{dy}{dx}=0$, the axis of x becomes a common tangent to both of

the branches: also, for $x=b+h$ the values of y are then impossible, and for $x=b-h$, there are two possible values of $\frac{d^2y}{dx^2}$, one + and the other −; whence it follows that the point B in this case becomes a point of reflection or cusp of the first species by (238).

If a be $<b$, then when $x=b$, we have $y=0$ and $\frac{dy}{dx}$ imaginary, and B will by (226) become an isolated or conjugate point.

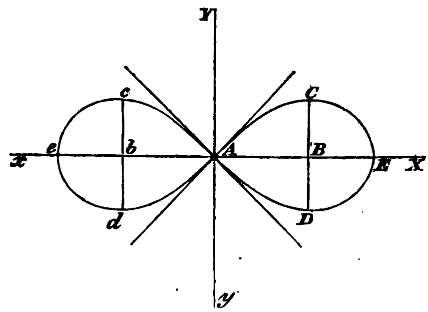
253. After a careful perusal of the examples here solved at considerable length, it can never be a matter of much difficulty to trace the curve defined by any equation that may be proposed, provided that equation can be solved with respect to either of the variables involved in it: and when this cannot be done, the principles laid down and exemplified in the preceding pages will be amply sufficient for the determination of every peculiarity that may belong to it. As this subject however is of considerable importance in Natural Philosophy, and has not been treated of at much length by any writer whose works are accessible to the generality of readers, there shall be added here a few more examples for the exercise of the Student with merely the diagrams and some particulars concerning them annexed.

Ex. 1. The equation is $y^2-2xy+3x^2-2y-4x+5=0$,



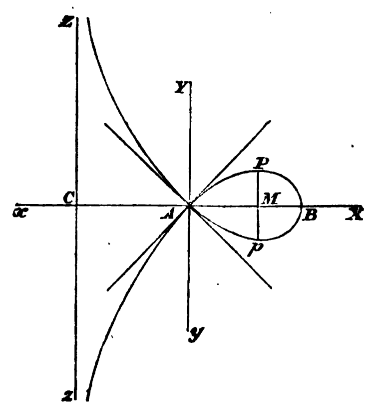
and the figure is an ellipse where $AB=1$, $BC=2$, $AD=2$ and $DE=3$.

Ex. 2. The equation is $(x^2+y^2)^2=2a^2(x^2-y^2)$, and the figure is



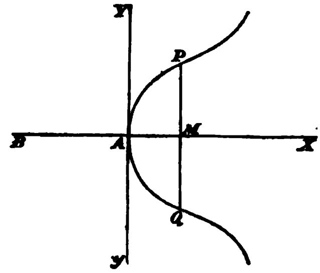
where $AE=Ae=a\sqrt{2}$, $AB=Ab=\frac{1}{2}a\sqrt{3}$ and $CBD=cbd=a$: also, the rectilineal tangents at the double point A are equally inclined to both the axes, and there are two inflexions there.

Ex. 3. The equation is $(a+x)y^2=(a-x)x^2$, and the figure is



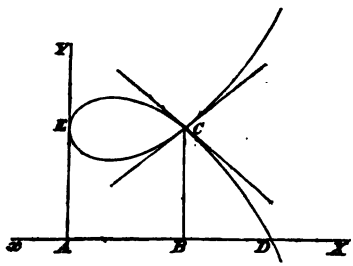
where $AB=AC=a$, and the rectilineal tangents at the double point A are equally inclined to both the co-ordinate axes.

Ex. 4. The equation is $ay^2 = x(x+b)^2$, and the figure is



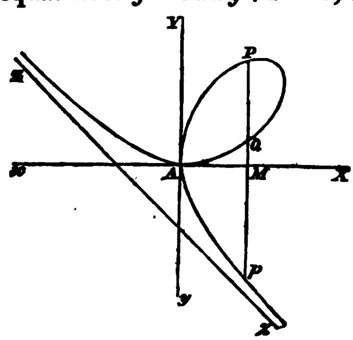
where if $AB = b$, B is a conjugate point, and if $AM = \frac{1}{3}b$, P and p are points of inflexion.

Ex. 5. The equation is $(y-b)^2 = x(x-a)^2$, and the figure is



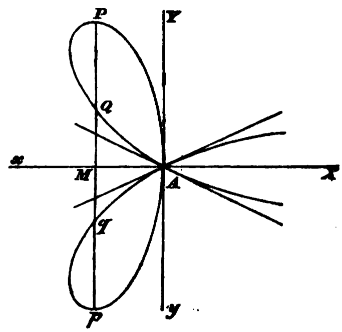
where $AB = a$, $BC = b$ and C is a double point.

Ex. 6. The equation is $y^3 - 3axy + x^3 = 0$, and the figure is



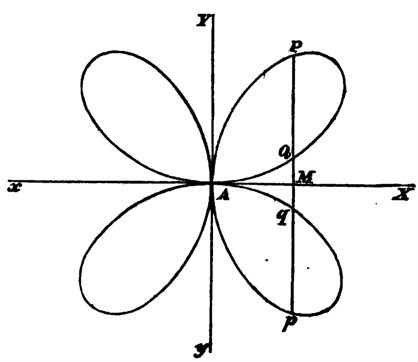
the limits of x being $a\sqrt[3]{4}$ and $\pm\infty$ and those of y being $a\sqrt[3]{4}$ and $\pm\infty$, and Zz is an asymptote equally inclined to the axes.

Ex. 7. The equation is $y^4 + 2axy^2 - ax^3 = 0$, and the figure is



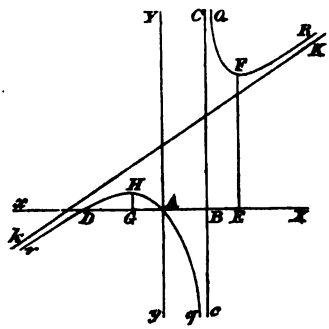
where the limits in the direction of the axis of x are ∞ and $-a$, and those in the direction of the axis of y are $\pm\infty$ and $\pm a$: there is moreover a triple point at the origin, one of the rectilineal tangents being the axis of y and the two others being inclined to the axis of x at angles whose tangents are $\pm\frac{1}{\sqrt{2}}$.

Ex. 8. The equation is $(x^2+y^2)^3 - 4a^2x^2y^2 = 0$, and the figure is



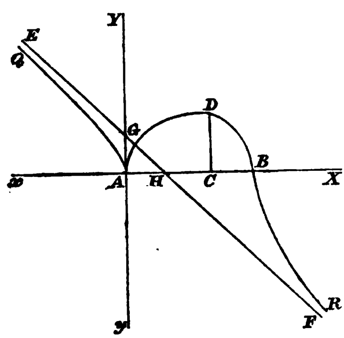
the limits of x being $\pm\frac{4a}{3\sqrt{3}}$, and those of y being $\pm\frac{4a}{3\sqrt{3}}$: and the origin is a quadruple point, the co-ordinate axes being the rectilineal tangents of the branches forming it.

Ex. 9. The equation is $(x-1)y=(x+1)x$, and the figure is



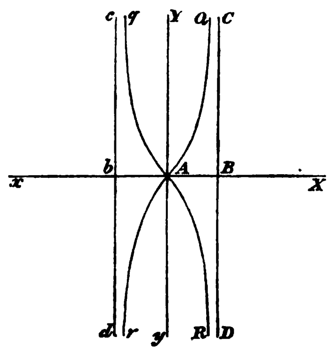
where $AB=1$, $AE=1+\sqrt{2}$, EF a minimum ordinate; $AG=1-\sqrt{2}$, GH a maximum ordinate; and CBc, Kk are asymptotes, the former being perpendicular to the axis of x and the latter inclined to it at 45^0: also, the curve cuts the axis of x in A and D at angles whose trigonometrical tangents are -1 and $\frac{1}{2}$, and that of y in A at an angle of 45^0.

Ex. 10. The equation is $y^3=ax^2-x^3$, and the figure is



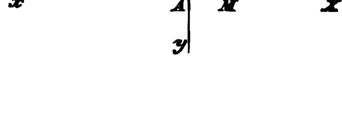
where $AB=a$; and if $AC=\frac{2}{3}a$, CD is a maximum ordinate: also, there is a cusp of the first species at A and a point of inflection at B, and the asymptote EHF makes with the axis of x an angle of 135^0.

Ex. 11. The equation is $(a^2 - x^2)\,y^2 = (a^2 + x^2)\,x^2$, and the figure is



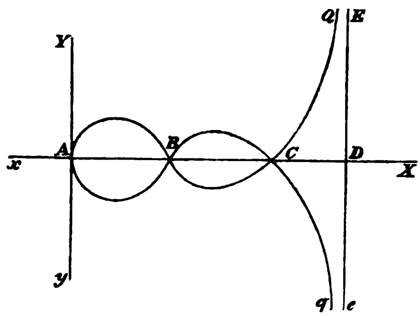
where $AB = Ab = a$ and CBD, cbd are asymptotes; also there is a point of inflexion of each branch at the origin where they cut the axis of x at 45^0 and 135^0 respectively, and moreover the origin is a double point of the first species.

Ex. 12. The equation is $x^4 - ax^2y + \frac{1}{5}a^2y^2 - ay^3 = 0$, and the figure is



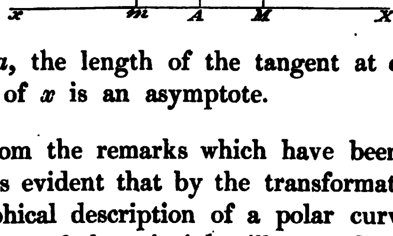
where $AB = \frac{1}{5}a$ and A is a double point of the second species. This curve is most easily traced by finding the value of x in terms of y.

Ex. 13. The equation is $x^2 + y^2 = \dfrac{b^2 x^2}{2ax - x^2}$, and the figure is



where $AB = a - \sqrt{a^2 - b^2}$, $AC = a + \sqrt{a^2 - b^2}$, $AD = 2a$ and EDe is an asymptote.

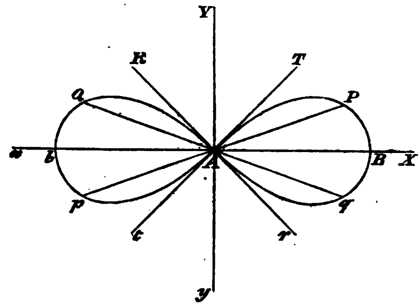
Ex. 14. The equation is $\dfrac{a + \sqrt{a^2 - y^2}}{y} = e^{\frac{x + \sqrt{a^2 - y^2}}{a}}$, and the figure is



where $AB = a$, the length of the tangent at every point $= a$ and the axis of x is an asymptote.

254. From the remarks which have been already made in (250), it is evident that by the transformation of co-ordinates, a graphical description of a polar curve may be obtained by means of the principles illustrated in the preceding examples. Such transformation however will be by no means necessary, as the polar equation is quite sufficient for the determination of the figure of the curve and its peculiarities, as will be seen by the following instance.

Ex. Trace the Lemniscata of *Bernoulli* whose polar equation is $r^2 = a^2 \cos 2\theta$.



Let xAX be the fixed axis from which the angle θ is measured, A the pole: then if $\theta = 0^0$, we have $r = \pm a$: hence if $AB = Ab = a$, the curve passes through B and b: if θ be $< 45^0$ we have $AP = Ap =$ a real magnitude, whence P and p are points in the curve: if $\theta = 45^0$, $r = 0$, therefore the curve passes through the pole A: if θ be $> 45^0$ and $< 135^0$, the values of r are imaginary and therefore no part of the curve lies between those values of the angle: if θ be $> 135^0$ and $< 180^0$ the values of r again become real, so that $AQ = Aq =$ a real magnitude, which determines the points Q and q in the curve: if $\theta = 180^0$, we have $r = \mp a$, which are represented by Ab, AB: and if θ be taken through the next semicircle the same arcs will be the locus of P: thus is the spiral described.

At any point P, the angle contained between the radius vector and rectilineal tangent is $\dfrac{rd\theta}{dr}$ by (192); and if ϕ denote this angle, we have here

$$\tan\phi = -\cot 2\theta:$$

hence if $\theta = 0$ or 180^0, $\tan\phi = \mp \infty$ or $\phi = \mp 90^0$, and therefore the tangents at B and b are perpendicular to the radius vector or to the fixed axis: if $\theta = 45^0$ or 135^0, we have $\phi = 0$, or the tangents coincide with the corresponding radii vectores: hence if the two indefinite lines TAt, RAr be drawn so as to make angles of 45^0 and 135^0 with the fixed

axis, they will be tangents to the curve at the pole which is therefore a double point.

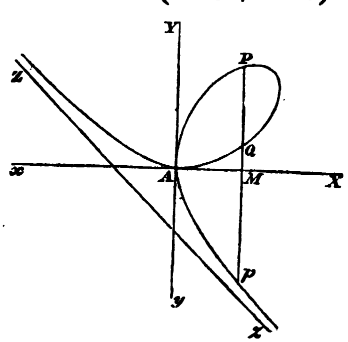
Also, $\dfrac{dr}{d\theta} = \pm \dfrac{a \sin 2\theta}{\sqrt{\cos 2\theta}} = 0$ gives $\sin 2\theta = 0$, from which we find $\theta = 0^0$ or $\theta = 180^0$: and therefore AB and Ab are maximum values of the radius vector.

Again, since $p = \dfrac{r^3}{a^2}$, we have here $\dfrac{dp}{dr} = \dfrac{3r^2}{a^2}$, which put $= 0$ gives $r = 0$, and this indicates the existence of a point of inflexion of each of the intersecting branches at the pole.

If in any case r remain finite when $\theta = \infty$, it is evident that the spiral admits of an asymptotic circle whose radius is the corresponding value of r; as when $r = \dfrac{a\theta^2}{\theta^2 - 1}$, the radius of this circle is a.

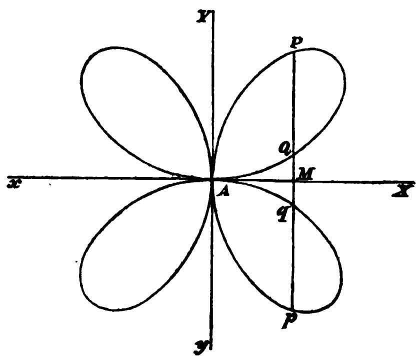
255. It frequently happens that the description of a curve may be obtained with facility from the polar equation, when that between the rectangular co-ordinates would lead to very complicated operations, as the student will readily discover in the following instances.

Ex. 1. If the rectangular equation be $y^3 - 3axy + x^3 = 0$, the polar equation is $r = \dfrac{3a \sin 2\theta}{2(\sin^3\theta + \cos^3\theta)}$; and the figure is



the lines xAX and YAy being tangents at A.

Ex. 2. The rectangular equation to the curve being $(x^2+y^2)^3=4a^2x^2y^2$ gives $r=a\sin 2\theta$ for the polar equation, and the figure is underneath.



Ex. 3. The rectangular equation to the *Cardioide* is $(x^2+y^2-ax)^2=a^2\{(x-a)^2+y^2\}$ and gives immediately $r=a(1+\cos\theta)$, by means of which the curve is easily traced.

CHAP. XII.

On the Developement of Functions of two or more independent Variables by means of the Theorems of Taylor and Maclaurin. On the Differentiation of Functions of two or more independent Variables. On the Elimination of indeterminate Functions whose particular forms are unknown. On the Theorem of Lagrange and its uses, and the Theorem of Laplace.

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256. Hitherto our attention has been confined to the consideration of functions of one independent or principal variable expressed generally by the equations $u = f(x)$ or $f(u, x) = 0$, according as they are explicit or implicit in their form: and we will now proceed to a more extensive view of the subject, wherein one quantity u is regarded as a function of two or more other quantities x, y, z, &c. entirely independent of each other, and is expressed explicitly or implicitly by the former or latter of the general equations

$$u = f(x, y, z, \&c.) \text{ or } f(u, x, y, z, \&c.) = 0.$$

I. NOTATION.

257. It is manifest that we shall now have occasion to make use of a Notation co-extensive with the changes that the function may undergo by the variation of each of the quantities upon which it depends, and an explanation of such Notation will be briefly attempted in the following article.

258. In the preceding pages we have considered $\dfrac{du}{dx}$ as representing the differential coefficient of u varying by a change

in x only, and therefore $\frac{du}{dy}$ will represent its differential coefficient relatively to y when y alone is variable: and thus the differentials of u arising from the same considerations are denoted by $\frac{du}{dx}dx$ and $\frac{du}{dy}dy$ respectively. These expressions are denominated the *Partial Differentials* of the function u relatively to the two independent variables x and y, and the *Total Differential* is that which is supposed to arise from the changes of both the independent variables taking place simultaneously, and is expressed by du and sometimes by $d(u)$.

Again, since $\frac{d\left(\frac{du}{dx}\right)}{dx}$ or $\frac{d^2u}{dx^2}$ and $\frac{d\left(\frac{du}{dy}\right)}{dy}$ or $\frac{d^2u}{dy^2}$ denote the second differential coefficients of u considered as a function of x and y respectively, we shall obviously have the differential coefficient of $\frac{du}{dx}$ considered as a function of y expressed by $\frac{d\left(\frac{du}{dx}\right)}{dy}$ or $\frac{d^2u}{dxdy}$: so of $\frac{du}{dy}$ treated as a function of x, the differential coefficient will be expressed by $\frac{d^2u}{dydx}$: that is, the second differential coefficient of u is expressed by $\frac{d^2u}{dxdy}$ or $\frac{d^2u}{dydx}$ according as we suppose the operation to be performed first with respect to x and then with respect to y or *vice versa*.

Moreover, the differential coefficient of $\frac{d^2u}{dxdy}$ regarded as a function of x will, on the same principle, be denoted by $\frac{d\left(\frac{d^2u}{dxdy}\right)}{dx}$, and may be written $\frac{d^3u}{dxdydx}$: this expression,

therefore, denotes the result of three differentiations of u, the first being made relatively to x, the second to y, and the third to x again. Similarly, the second differential coefficient of $\frac{du}{dx}$ relatively to y will be expressed by $\frac{d^2\left(\frac{du}{dx}\right)}{dy^2}$ or $\frac{d^3u}{dx\,dy^2}$, which denotes the result of three differentiations of u, the first with respect to x and the last two with respect to y.

Also, if after having found the m^{th} differential coefficient of u considered as a function of x, we take the n^{th} differential coefficient of the result regarded as a function of y, we shall express the effect of all these operations by

$$\frac{d^n\left(\frac{d^m u}{dx^m}\right)}{dy^n} \text{ or } \frac{d^{m+n}u}{dx^m\,dy^n}.$$

Precisely in the same manner, when there are more independent variables x, y, z, &c., the expression $\frac{d^{l+m+n+\&c.}u}{dx^l\,dy^m\,dz^n\,\&c.}$ will shew that the function is understood to have been differentiated $l+m+n+$ &c. times: first, l times with respect to x; next, m times with respect to y; then n times with respect to z, and so on.

Another kind of Notation attended with some conveniences has of late been partially adopted. In this, the differential coefficients are no longer expressed in a fractional form, but are denoted by the letter d with the principal variables suffixed: thus, $\frac{du}{dx}$ and $\frac{du}{dy}$ are equivalent to $d_x u$ and $d_y u$: so $\frac{d^2u}{dx\,dy}$ and $\frac{d^2u}{dy\,dx}$ are equivalent to $d_y d_x u$ and $d_x d_y u$, and $\frac{d^{l+m+n+\&c.}u}{dx^l dy^m dz^n\,\&c.}$ may be written $d_x^{\,l} d_y^{\,m} d_z^{\,n}$ &c. u: and though we shall at present adhere to the former system, there never can exist the least difficulty in passing from the one to the other.

II. DEVELOPEMENT OF A FUNCTION OF TWO VARIABLES.

259. *Given* $u=f(x, y)$, *to find the developement of* $u'=f(x+h, y+k)$, h *and* k *being any two indeterminate magnitudes.* u ι

Here it is obvious that the final result will be the same whether we suppose the changes in x and y to take place simultaneously, or first in one of them and then in the other: on the latter hypothesis let u be changed into $'u$ whilst x becomes $x+h$ and y remains unchanged: then, by *Taylor's* Theorem, we shall have

$$'u=f(x+h, y)$$

$$=u+\frac{du}{dx}h+\frac{d^2u}{dx^2}\frac{h^2}{1.2}+\frac{d^3u}{dx^3}\frac{h^3}{1.2.3}+\&c.:$$

now if y be supposed to become $y+k$ whilst x undergoes no further change, it is evident that on this account,

(1) u will be changed into

$$u+\frac{du}{dy}k+\frac{d^2u}{dy^2}\frac{k^2}{1.2}+\frac{d^3u}{dy^3}\frac{k^3}{1.2.3}+\&c.:$$

(2) $\frac{du}{dx}$ will be changed into

$$\frac{du}{dx}+\frac{d\left(\frac{du}{dx}\right)}{dy}k+\frac{d^2\left(\frac{du}{dx}\right)}{dy^2}\frac{k^2}{1.2}+\frac{d^3\left(\frac{du}{dx}\right)}{dy^3}\frac{k^3}{1.2.3}+\&c.$$

$$=\frac{du}{dx}+\frac{d^2u}{dxdy}k+\frac{d^3u}{dxdy^2}\frac{k^2}{1.2}+\frac{d^4u}{dxdy^3}\frac{k^3}{1.2.3}+\&c.:$$

(3) $\dfrac{d^2u}{dx^2}$ will be changed into

$$\frac{d^2u}{dx^2}+\frac{d\left(\dfrac{d^2u}{dx^2}\right)}{dy}k+\frac{d^2\left(\dfrac{d^2u}{dx^2}\right)}{dy^2}\frac{k^2}{1.2}+\frac{d^3\left(\dfrac{d^2u}{dx^2}\right)}{dy^3}\frac{k^3}{1.2.3}+\&c.$$

$$=\frac{d^2u}{dx^2}+\frac{d^3u}{dx^2dy}k+\frac{d^4u}{dx^2dy^2}\frac{k^2}{1.2}+\frac{d^5u}{dx^2dy^3}\frac{k^3}{1.2.3}+\&c.:$$

(4) $\dfrac{d^3u}{dx^3}$ will be changed into

$$\frac{d^3u}{dx^3}+\frac{d\left(\dfrac{d^3u}{dx^3}\right)}{dy}k+\frac{d^2\left(\dfrac{d^3u}{dx^3}\right)}{dy^2}\frac{k^2}{1.2}+\frac{d^3\left(\dfrac{d^3u}{dx^3}\right)}{dy^3}\frac{k^3}{1.2.3}+\&c.$$

$$=\frac{d^3u}{dx^3}+\frac{d^4u}{dx^3dy}k+\frac{d^5u}{dx^3dy^2}\frac{k^2}{1.2}+\frac{d^6u}{dx^3dy^3}\frac{k^3}{1.2.3}+\&c.:$$

and so on, for the succeeding terms: whence by substitution, since $'u$ will now be changed into u', we shall have

$$u'=f(x+h,y+k)$$

$$=u+\frac{du}{dy}k+\frac{d^2u}{dy^2}\frac{k^2}{1.2}+\frac{d^3u}{dy^3}\frac{k^3}{1.2.3}+\&c.:$$

$$+\frac{du}{dx}h+\frac{d^2u}{dx\,dy}kh+\frac{d^3u}{dx\,dy^2}\frac{k^2h}{1.2}+\&c.$$

$$+\frac{d^2u}{dx^2}\frac{h^2}{1.2}+\frac{d^3u}{dx^2dy}\frac{kh^2}{1.2}+\&c.$$

$$+\frac{d^3u}{dx^3}\frac{h^3}{1.2.3}+\&c.$$

$$+\&c.$$

which is in fact the expression of *Taylor's* Theorem when applied to a function of the two independent variables x and y.

260. Cor. Since the general term of the developement of $'u = f(x+h, y)$ is $\dfrac{d^m u}{dx^m}\dfrac{h^m}{1.2.3.\&c.m}$, and this becomes

$$\left\{\frac{d^m u}{dx^m} + \frac{d^{m+1}u}{dx^m dy}k + \frac{d^{m+2}u}{dx^m dy^2}\frac{k^2}{1.2} + \&c. + \frac{d^{m+n}u}{dx^m dy^n}\frac{k^n}{1.2.3.\&c.n} + \&c.\right\}$$

$\times \dfrac{h^m}{1.2.3.\&c.m}$ when y is changed into $y+k$, it is manifest that the general term of the developement of u' will be

$$\frac{d^{m+n}u}{dx^m dy^n}\frac{h^m k^n}{(1.2.3.\&c.m)(1.2.3.\&c.n)},$$

from which every term may be obtained by assigning to m and n all possible positive integral values in succession.

This general term may be likewise written in the form

$$\frac{d^{m+n}u}{dx^m dy^n}\frac{(m+n)(m+n-1)\&c.(m+1)h^m k^n}{\{1.2.3.\&c.(m+n)\}(1.2.3.\&c.n)}$$

$$= \frac{d^{m+n}u}{dx^m dy^n}\frac{1}{1.2.3.\&c.(m+n)}\frac{(m+n)(m+n-1)\&c.(m+1)h^m k^n}{1.2.3.\&c.n}$$

$= \dfrac{1}{1.2.3.\&c.(m+n)}\dfrac{d^{m+n}u}{dx^m dy^n}$ multiplied into the $(n+1)^{\text{th}}$ term of the expansion of $(h+k)^{m+n}$: and the developement may be exhibited also in the form,

$$u' = f(x+h, y+k) =$$

$$\begin{aligned}
&u \\
&+ \frac{1}{1}\left(\frac{du}{dx}h + \frac{du}{dy}k\right) \\
&+ \frac{1}{1.2}\left(\frac{d^2u}{dx^2}h^2 + 2\frac{d^2u}{dx\,dy}hk + \frac{d^2u}{dy^2}k^2\right) \\
&+ \frac{1}{1.2.3}\left(\frac{d^3u}{dx^3}h^3 + 3\frac{d^3u}{dx^2\,dy}h^2k + 3\frac{d^3u}{dx\,dy^2}hk^2 + \frac{d^3u}{dy^3}k^3\right) \\
&+ \&c. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Ex. Given $u=x^m y^n$ and let it be required to find the developement of $u'=(x+h)^m(y+k)^n$.

Here we have $u=x^m y^n$, whence are obtained

$$\frac{du}{dx}=mx^{m-1}y^n,\quad \frac{du}{dy}=nx^m y^{n-1};$$

$$\frac{d^2u}{dx^2}=m(m-1)x^{m-2}y^n,\quad \frac{d^2u}{dx\,dy}=mnx^{m-1}y^{n-1},$$

$$\frac{d^2u}{dy^2}=n(n-1)x^m y^{n-2};$$

$$\frac{d^3u}{dx^3}=m(m-1)(m-2)x^{m-3}y^n,$$

$$\frac{d^3u}{dx^2\,dy}=mn(m-1)x^{m-2}y^{n-1},$$

$$\frac{d^3u}{dx\,dy^2}=mn(n-1)x^{m-1}y^{n-2},$$

$$\frac{d^3u}{dy^3}=n(n-1)(n-2)x^m y^{n-3};$$

and so on: therefore by the substitution of these values in the general formula, we shall have

$$(x+h)^m(y+k)^n=$$

$$x^m y^n+mx^{m-1}y^n h+\frac{m(m-1)}{1.2}x^{m-2}y^n h^2+\&c.$$

$$+nx^m y^{n-1}k+\frac{mn}{1.1}x^{m-1}y^{n-1}hk\quad+\&c.$$

$$+\frac{n(n-1)}{1.2}x^m y^{n-2}k^2+\&c.$$

$$+\&c.$$

which might have been obtained also by expanding each of the binomials $(x+h)^m$ and $(y+k)^n$ and multiplying together the two results, or by substituting $y+k$ for y in the developement of $(x+h)^m y^n$.

261. In the preceding proposition it has been supposed that x is first changed into $x+h$ and then y into $y+k$, to ascertain the developement of $u'=f(x+h,\ y+k)$; but if the order of these substitutions be changed, it is manifest that the same result ought to be obtained. The operations being performed first with respect to $y+k$ and then with regard to $x+h$ in the same manner lead to

$$\begin{aligned} u'=f(x+h,\ y+k)=u+\frac{du}{dx}h+\frac{d^2u}{dx^2}\frac{h^2}{1.2}+\frac{d^3u}{dx^3}\frac{h^3}{1.2.3}+\&c. \\ +\frac{du}{dy}k+\frac{d^2u}{dy\,dx}kh+\frac{d^3u}{dy\,dx^2}\frac{kh^2}{1.2}+\&c. \\ +\frac{d^2u}{dy^2}\frac{k^2}{1.2}+\frac{d^3u}{dy^2\,dx}\frac{k^2h}{1.2}+\&c. \\ +\frac{d^3u}{dy^3}\frac{k^3}{1.2.3}+\&c. \\ +\&c., \end{aligned}$$

which being identical with the one deduced on the other hypothesis, we may obviously equate the coefficients of the same powers and combinations of h and k in the two expressions: whence we shall have

$$\frac{d^2u}{dy\,dx}=\frac{d^2u}{dx\,dy},$$

$$\frac{d^3u}{dy\,dx^2}=\frac{d^3u}{dx^2\,dy},$$

$$\frac{d^3u}{dy^2\,dx}=\frac{d^3u}{dx\,dy^2};$$

&c................

$$\frac{d^{m+n}u}{dy^m\,dx^n}=\frac{d^{m+n}u}{dx^n\,dy^m}:$$

and indeed if the first of these results be once established, we have by means of the notation adopted,

$$\frac{d^3u}{dy\,dx^2}=\frac{d^3u}{dy\,dx\,dx}=\frac{d\left(\dfrac{d^2u}{dy\,dx}\right)}{dx}=\frac{d\left(\dfrac{d^2u}{dx\,dy}\right)}{dx}=\frac{d^3u}{dx\,dy\,dx};$$

and so on;

that is, in finding the partial differential coefficients of any function, the order in which the operations are performed will not affect the result.

Ex. 1. Let $u=x^m y^n$; then we shall have

$$\frac{du}{dx}=my^n x^{m-1},\ \therefore\ \frac{d^2u}{dx\,dy}=mnx^{m-1}y^{n-1},$$

$$\text{and } \frac{du}{dy}=nx^m y^{n-1},\ \therefore\ \frac{d^2u}{dy\,dx}=mnx^{m-1}y^{n-1};$$

whence it follows that

$$\frac{d^2u}{dx\,dy}=\frac{d^2u}{dy\,dx}:$$

again, from the result $\dfrac{d^2u}{dx\,dy}=mnx^{m-1}y^{n-1}$,

we get

$$\frac{d^3u}{dx\,dy\,dx}=m(m-1)ny^{n-1}x^{m-2}:$$

also, from $\dfrac{d^2u}{dy\,dx}=mnx^{m-1}y^{n-1}$,

we obtain

$$\frac{d^3u}{dy\,dx^2}=m(m-1)ny^{n-1}x^{m-2}:$$

so that we have $\dfrac{d^3u}{dx\,dy\,dx}=\dfrac{d^3u}{dy\,dx^2}$, and so on.

Ex. 2. Let $u = x \log y$: whence are obtained immediately

$$\frac{du}{dx} = \log y \text{ and } \frac{d^2u}{dx\,dy} = \frac{1}{y}:$$

$$\frac{du}{dy} = \frac{x}{y} \text{ and } \frac{d^2u}{dy\,dx} = \frac{1}{y};$$

$$\text{so that } \frac{d^2u}{dx\,dy} = \frac{d^2u}{dy\,dx}:$$

again, from $\dfrac{d^2u}{dx\,dy} = \dfrac{1}{y}$, we find $\dfrac{d^3u}{dx\,dy\,dx} = 0 = \dfrac{d^3u}{dy\,dx^2}$ as deducible from $\dfrac{d^2u}{dy\,dx} = \dfrac{1}{y}$.

262. If we denote the values of u and its successive differential coefficients taken in the order in which they occur in the formula of (260) by

$$A_1;\ B_1,\ B_2;\ C_1,\ C_2,\ C_3;\ D_1,\ D_2,\ D_3,\ D_4;\ \&c.$$

when $x = 0$ and $y = 0$, the theorem becomes

$$u' = f(h,\ k) =$$

$$A_1 + \frac{1}{1}(B_1h + B_2k) + \frac{1}{1\,.\,2}(C_1h^2 + 2C_2hk + C_3k^2)$$
$$+ \frac{1}{1\,.\,2\,.\,3}(D_1h^3 + 3D_2h^2k + 3D_3hk^2 + D_4k^3) + \&c.:$$

whence substituting x and y in the places of h and k and therefore u in the place of u', we find

$$u = f(x,\ y) =$$
$$A_1$$
$$+ \frac{1}{1}(B_1x + B_2y)$$
$$+ \frac{1}{1\,.\,2}(C_1x^2 + 2C_2xy + C_3y^2)$$
$$+ \frac{1}{1\,.\,2\,.\,3}(D_1x^3 + 3D_2x^2y + 3D_3xy^2 + Dy^3)$$
$$+ \&c.$$

which is *Maclaurin's* Theorem applied to the developement of a function of two variables x, y, and is deduced from *Taylor's* exactly as has been done for the case of one variable in (85).

III. DIFFERENTIATION OF FUNCTIONS OF TWO VARIABLES.

263. *To find the successive Differentials of a Function of two Variables expressed generally by $u = f(x, y)$.*

We have already proved in the preceding articles that

$$u' =$$

$$\begin{aligned}
u &+ \frac{du}{dx}h + \frac{d^2u}{dx^2}\,\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\,\frac{h^3}{1.2.3} + \&c. \\
&+ \frac{du}{dy}k + \frac{d^2u}{dx\,dy}\,\frac{hk}{1.1} + \frac{d^3u}{dx^2\,dy}\,\frac{h^2k}{1.2.1} + \&c. \\
&\qquad + \frac{d^2u}{dy^2}\,\frac{k^2}{1.2} + \frac{d^3u}{dx\,dy^2}\,\frac{hk^2}{1.1.2} + \&c. \\
&\qquad\qquad + \frac{d^3u}{dy^3}\,\frac{k^3}{1.2.3} + \&c. \\
&\qquad\qquad\qquad + \&c.:
\end{aligned}$$

whence adopting the system of *Lagrange* as explained in (95) and the subsequent articles, and expressing the indeterminate magnitudes h and k symbolically by dx and dy, we shall have the following equations:

(1) $\quad du = \dfrac{du}{dx}dx + \dfrac{du}{dy}dy;$

(2) $\quad d^2u = \dfrac{d^2u}{dx^2}dx^2 + 2\dfrac{d^2u}{dx\,dy}dx\,dy + \dfrac{d^2u}{dy^2}dy^2;$

(3) $\quad d^3u = \dfrac{d^3u}{dx^3}dx^3 + 3\dfrac{d^3u}{dx^2\,dy}dx^2\,dy + 3\dfrac{d^3u}{dx\,dy^2}dx\,dy^2 + \dfrac{d^3u}{dy^3}dy^3;$

&c...

which are the first, second, third, &c. differentials of the function $u=f(x, y)$, and the law of the formation of their different terms is that explained in (260).

264. Cor. 1. From the first result obtained in the last article, we conclude that the *Total Differential* of a function of two variables is the sum of its *Partial Differentials*, which in the Notation adopted is

$$du=\frac{du}{dx}dx+\frac{du}{dy}dy \text{ or } d(u)=\frac{du}{dx}dx+\frac{du}{dy}dy:$$

and this will enable us to obtain with great facility the differentials of succeeding orders, recollecting that dx and dy are here considered of invariable magnitude: thus,

$$d^2u=d\left\{\frac{du}{dx}dx\right\}+d\left\{\frac{du}{dy}dy\right\};$$

$$\text{but } d\left(\frac{du}{dx}\right)=\frac{d^2u}{dx^2}dx+\frac{d^2u}{dx\,dy}dy,$$

$$\text{and } d\left(\frac{du}{dy}\right)=\frac{d^2u}{dy^2}dy+\frac{d^2u}{dx\,dy}dx;$$

$$\therefore\ d^2u=\frac{d^2u}{dx^2}dx^2+2\frac{d^2u}{dx\,dy}dx\,dy+\frac{d^2u}{dy^2}dy^2:$$

and from this last is obtained in the same manner

$$d^3u=\frac{d^3u}{dx^3}dx^3+3\frac{d^3u}{dx^2\,dy}dx^2dy+3\frac{d^3u}{dx\,dy^2}dx\,dy^2+\frac{d^3u}{dy^3}dy^3:$$

and so on.

265. Cor. 2. It is manifest from what has been said already that if we take h and k in (260) to be represented in magnitude by the indeterminate quantities dx and dy, we shall have the formula

$$u'=u+\frac{1}{1}du+\frac{1}{1.2}d^2u+\frac{1}{1.2.3}d^3u+\&c.,$$

already proved for a function of one variable in (81) and by means of which, as in (82), the general differential of a function of two variables may be deduced, provided the general term of the developement of the function can be ascertained.

266. Cor. 3. If in $u=f(x, y)$ we assume $f(x, y)=f(t)$ it is obvious that $du=f'(t)\,dt$ where $f'(t)$ is a new function of t as appears in the differentiation of functions of one variable: whence the partial differential coefficients relative to x and y will be respectively

$$\frac{du}{dx}=f'(t)\frac{dt}{dx} \text{ and } \frac{du}{dy}=f'(t)\frac{dt}{dy},$$

the factor $f'(t)$ or $f'(x, y)$ being the same for both.

267. We will now exemplify what has been said in the preceding articles by a few instances.

Ex. 1. Let $u=x^3+ax^2y+y^3$; then we immediately find the partial differentials to be

$$\frac{du}{dx}dx=(3x^2+2axy)\,dx,$$

$$\frac{du}{dy}dy=(ax^2+3y^2)\,dy;$$

whence the total differential will be

$$du=(3x^2+2axy)\,dx+(ax^2+3y^2)\,dy:$$

and the succeeding differentials may be found in the same manner.

Ex. 2. Let $u=\frac{x}{y}+\frac{y}{x}$; then the partial differential coefficients will be

$$\frac{du}{dx}=\frac{1}{y}-\frac{y}{x^2} \text{ and } \frac{du}{dy}=\frac{1}{x}-\frac{x}{y^2}:$$

$$\text{whence } du = \frac{du}{dx}dx + \frac{du}{dy}dy$$

$$= \frac{dx}{y} - \frac{ydx}{x^2} + \frac{dy}{x} - \frac{xdy}{y^2}.$$

Ex. 3. Let $u = \log\left(\frac{x+\sqrt{x^2-y^2}}{y}\right)$

$$= \log(x+\sqrt{x^2-y^2}) - \log y:$$

$$\therefore \frac{du}{dx} = \frac{x}{\sqrt{x^2-y^2}} \text{ and } \frac{du}{dy} = -\frac{x}{y\sqrt{x^2-y^2}};$$

$$\text{whence } du = \frac{du}{dx}dx + \frac{du}{dy}dy$$

$$= \frac{dx}{\sqrt{x^2-y^2}} - \frac{xdy}{y\sqrt{x^2-y^2}} = \frac{ydx - xdy}{y\sqrt{x^2-y^2}}.$$

Ex. 4. Let $u = \log\tan\frac{x}{y}$: then the coefficients are

$$\frac{du}{dx} = \frac{d\tan\frac{x}{y}}{dx\tan\frac{x}{y}} = \frac{\sec^2\frac{x}{y}}{y\tan\frac{x}{y}}, \quad \frac{du}{dy} = \frac{d\tan\frac{x}{y}}{dy\tan\frac{x}{y}} = \frac{-\sec^2\frac{x}{y}}{y^2\tan\frac{x}{y}}$$

$$\therefore du = \frac{du}{dx}dx + \frac{du}{dy}dy = \frac{\sec^2\frac{x}{y}}{y\tan\frac{x}{y}}dx - \frac{x\sec^2\frac{x}{y}}{y^2\tan\frac{x}{y}}dy$$

$$= \frac{ydx - xdy}{y^2\tan\frac{x}{y}}\sec^2\frac{x}{y} = \frac{ydx - xdy}{y^2\sin\frac{x}{y}\cos\frac{x}{y}}:$$

and the differentials of succeeding orders may be found by a similar process, dx and dy being constant quantities.

Ex. 5. Let $u = xy\phi(y)$, where ϕ represents any function whatever:

$$\text{then } \frac{du}{dx} = y\phi(y) \text{ and } \frac{du}{dy} = x\phi y + xy\frac{d\phi(y)}{dy};$$

and denoting $\dfrac{d\phi(y)}{dy}$ by $\phi'(y)$ we shall have

$$\frac{du}{dy} = x\phi(y) + xy\phi'(y):$$

whence we find the total differential of this function to be

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy$$

$$= y\phi(y)\,dx + x\phi(y)\,dy + xy\phi'(y)\,dy.$$

Ex. 6. Let $u = \dfrac{x}{y}\phi(xy)$: then we have by means of (266),

$$\frac{du}{dx} = \frac{1}{y}\phi(xy) + \frac{x}{y}\phi'(xy)y$$

$$= \frac{1}{y}\phi(xy) + x\phi'(xy),$$

$$\frac{du}{dy} = -\frac{x}{y^2}\phi(xy) + \frac{x}{y}\phi'(xy)\,x$$

$$= -\frac{x}{y^2}\phi(xy) + \frac{x^2}{y}\phi'(xy):$$

so that the total differential of the proposed function is

$$du = \frac{1}{y}\phi(xy)\,dx + x\phi'(xy)\,dx - \frac{x}{y^2}\phi(xy)\,dy + \frac{x^2}{y}\phi'(xy)\,dy.$$

268. If $u=f(x, y)=0$, it is manifest that the variables x and y become dependent upon each other; and

$$u' = u + \frac{du}{dx}h + \frac{du}{dy}k + \&c. = 0:$$

but y being regarded as a function of x by virtue of the proposed equation, we have

$$k = \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \&c.:$$

$$\therefore\ u' = u + \frac{du}{dx}h + \frac{du}{dy}\left(\frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \&c.\right)$$

$$= u + \left(\frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx}\right)h + \&c.:$$

whence we have immediately as in (263)

$$du = \left(\frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx}\right)h \text{ or } du = \frac{du}{dx}dx + \frac{du}{dy}\frac{dy}{dx}dx:$$

and the corresponding differential coefficient will be expressed by

$$\frac{d\,(u)}{dx} = \frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx} = 0.$$

IV. DEVELOPEMENT AND DIFFERENTIATION OF FUNCTIONS OF THREE OR MORE INDEPENDENT VARIABLES.

269. *To find the developement of a function* u *of three variables* x, y, z *independent of each other.*

Here $u=f(x, y, z)$ and it is required to find the developement of $u'=f(x+h, y+k, z+l)$, where h, k and l are indeterminate magnitudes either positive or negative: then if we suppose the changes to take place separately, as we have done in (259) and denote by $''u$ the value of the function arising from the first two changes only, we shall have

$$''u = f(x+h,\, y+k,\, z) =$$

$$u$$

$$+ \frac{1}{1}\left(\frac{du}{dx}h + \frac{du}{dy}k\right)$$

$$+ \frac{1}{1\,.\,2}\left(\frac{d^2u}{dx^2}h^2 + 2\frac{d^2u}{dx\,dy}hk + \frac{d^2u}{dy^2}k^2\right)$$

$$+ \frac{1}{1\,.\,2\,.\,3}\left(\frac{d^3u}{dx^3}h^3 + 3\frac{d^3u}{dx^2\,dy}h^2k + 3\frac{d^3u}{dx\,dy^2}hk^2 + \frac{d^3u}{dy^3}k^3\right)$$

$$+ \&c. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

in both sides of which if z be changed into $z + l$,

(1) u will be changed into

$$u + \frac{du}{dz}l + \frac{d^2u}{dz^2}\,\frac{l^2}{1\,.\,2} + \frac{d^3u}{dz^3}\,\frac{l^3}{1\,.\,2\,.\,3} + \&c.:$$

(2) $\dfrac{du}{dx}$ will be changed into

$$\frac{du}{dx} + \frac{d^2u}{dx\,dz}l + \frac{d^3u}{dx\,dz^2}\,\frac{l^2}{1\,.\,2} + \frac{d^4u}{dx\,dz^3}\,\frac{l^3}{1\,.\,2\,.\,3} + \&c.:$$

&c...

(3) $\dfrac{du}{dy}$ will be changed into

$$\frac{du}{dy} + \frac{d^2u}{dy\,dz}l + \frac{d^3u}{dy\,dz^2}\,\frac{l^2}{1\,.\,2} + \frac{d^4u}{dy\,dz^3}\,\frac{l^3}{1\,.\,2\,.\,3} + \&c.:$$

&c...

and thence we have $u' = f(x+h,\, y+k,\, z+l)$

$$= u + \frac{du}{dz}l + \frac{d^2u}{dz^2}\,\frac{l^2}{1\,.\,2} + \frac{d^3u}{dz^3}\,\frac{l^3}{1\,.\,2\,.\,3} + \&c.$$

$$+h\left(\frac{du}{dx}+\frac{d^2u}{dx\,dz}l+\frac{d^3u}{dx\,dz^2}\frac{l^2}{1.2}+\&c.\right)$$

$$+k\left(\frac{du}{dy}+\frac{d^2u}{dy\,dz}l+\frac{d^3u}{dy\,dz^2}\frac{l^2}{1.2}+\&c.\right)$$

$$+\frac{h^2}{1.2}\left(\frac{d^2u}{dx^2}+\frac{d^3u}{dx^2\,dz}l+\frac{d^4u}{dx^2\,dz^2}\frac{l^2}{1.2}+\&c.\right)$$

$$+hk\left(\frac{d^2u}{dx\,dy}+\frac{d^3u}{dx\,dy\,dz}l+\frac{d^4u}{dx\,dy\,dz^2}\frac{l^2}{1.2}+\&c.\right)$$

$$+\frac{k^2}{1.2}\left(\frac{d^2u}{dy^2}+\frac{d^3u}{dy^2\,dz}l+\frac{d^4u}{dy^2\,dz^2}\frac{l^2}{1.2}+\&c.\right)$$

$$+\&c.\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

which is easily arranged according to the dimensions of h, k and l, if it be required.

Thus, a function of three independent variables may be developed by *Taylor's* Theorem in terms of the variables and their indeterminate increments: and by making $x=0$, $y=0$, $z=0$ as in (262), the expansion of $u=f(x, y, z)$ may be obtained from the Theorem of *Maclaurin*.

270. Cor. Hence if a proposed function of three variables as $u=f(x, y, z)$ be required to be differentiated, we see directly that

$$du=\frac{du}{dx}h+\frac{du}{dy}k+\frac{du}{dz}l$$

$$=\frac{du}{dx}dx+\frac{du}{dy}dy+\frac{du}{dz}dz,$$

if the indeterminate magnitudes h, k, l be supposed to be represented by dx, dy and dz as in (263): and from these the differentials of all the succeeding orders may be deduced according to the same principles.

271. The mode of proceeding here adopted may obviously be applied to all functions whatever be the number of independent variables employed, whether it be to find their developements according to *Taylor's* and *Maclaurin's* Theorems, or to determine their differentials: and it may in every case be similarly shewn that the *Total Differential* is equal to the sum of all the *Partial Differentials.*

Ex. 1. Let $u = \dfrac{xy^2}{z^2 - a^2}$: then first finding the partial differential coefficients we have

$$\frac{du}{dx} = \frac{y^2}{z^2 - a^2}, \quad \frac{du}{dy} = \frac{2xy}{z^2 - a^2}, \quad \frac{du}{dz} = -\frac{2xy^2z}{(z^2 - a^2)^2}:$$

therefore the total differential will be

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz$$

$$= \frac{y^2dx}{z^2 - a^2} + \frac{2xydy}{z^2 - a^2} - \frac{2xy^2zdz}{(z^2 - a^2)^2}$$

$$= \frac{y^2(z^2 - a^2)dx + 2xy(z^2 - a^2)dy - 2xy^2zdz}{(z^2 - a^2)^2}.$$

Ex. 2. Let $u = x\phi\left(\dfrac{x}{y}, \dfrac{y}{z}\right)$: then we find

$$\frac{du}{dx} = \phi\left(\frac{x}{y}, \frac{y}{z}\right) + \frac{x}{y}\phi'\left(\frac{x}{y}, \frac{y}{z}\right),$$

$$\frac{du}{dy} = -\frac{x^2}{y^2}\phi'\left(\frac{x}{y}, \frac{y}{z}\right) + \frac{x}{z}\phi'\left(\frac{x}{y}, \frac{y}{z}\right),$$

$$\frac{du}{dz} = -\frac{xy}{z^2}\phi'\left(\frac{x}{y}, \frac{y}{z}\right):$$

and therefore the total differential will be

$$du = \phi\left(\frac{x}{y}, \frac{y}{z}\right) dx + \frac{x}{y}\phi'\left(\frac{x}{y}, \frac{y}{z}\right) dx - \frac{x^2}{y^2}\phi'\left(\frac{x}{y}, \frac{y}{z}\right) dy$$

$$+ \frac{x}{z}\phi'\left(\frac{x}{y}, \frac{y}{z}\right) dy - \frac{xy}{z^2}\phi'\left(\frac{x}{y}, \frac{y}{z}\right) dz.$$

272. A *Homogeneous* function of two or more independent variables is connected with its differential coefficients in a peculiar manner, which may here be shortly explained.

Let u represent any homogeneous function of x, y, z, &c., where m is the sum of the exponents of the variables in each term: then if in the places of x, y, z, &c. there be substituted $(1+a)x$, $(1+a)y$, $(1+a)z$, &c. it is evident that u will become $(1+a)^m u$: but in the formula of (269), if ax, ay, az, &c. be put for h, k, l, &c., we shall obviously have

$$(1+a)^m u =$$

$$u + \frac{du}{dx} ax + \frac{du}{dy} ay + \text{\&c.}$$

$$+ \frac{1}{1.2}\left\{\frac{d^2u}{dx^2} a^2x^2 + 2\frac{d^2u}{dx\,dy} a^2xy + \frac{d^2u}{dy^2} a^2y^2 + \text{\&c.}\right\}$$

$$+ \text{\&c.} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

whence developing the former member by the binomial theorem and equating the coefficients of the same powers of a in both sides, we obtain the following results:

$$(1) \qquad mu = \frac{du}{dx} x + \frac{du}{dy} y + \frac{du}{dz} z + \text{\&c.}$$

$$(2) \qquad m(m-1)u = \frac{d^2u}{dx^2} x^2 + 2\frac{d^2u}{dx\,dy} xy + \frac{d^2u}{dy^2} y^2 + \text{\&c.}$$

$$\text{\&c.} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

which may easily be verified by taking any instance whatever of a homogeneous function.

273. Let $u = f(x, y, z) = 0$ be an equation betweerr the three variables x, y, z, wherein z may be considered as a function of the two independent variables x, y: then by (268) we have

$$\frac{du}{dx} + \frac{du}{dz}\frac{dz}{dx} = 0,$$

$$\frac{du}{dy} + \frac{du}{dz}\frac{dz}{dy} = 0:$$

whence multiplying the first of these by dx and the second by dy, we have by addition

$$\frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}\left(\frac{dz}{dx}dx + \frac{dz}{dy}dy\right) = 0:$$

but $dz = \frac{dz}{dx}dx + \frac{dz}{dy}dy$ by (263), whence by substitution we get

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz = 0:$$

that is, the first differential of $u = 0$ taken with respect to all the three variables must be equated to zero: and the total differential of the second order may be obtained by differentiating this last result, wherein z is a function of x and y and dx, dy are constant, though the same might easily be found by differentiating each of the terms of the equations

$$\frac{d(u)}{dx} = \frac{du}{dx} + \frac{du}{dz}\frac{dz}{dx} = 0,$$

$$\frac{d(u)}{dy} = \frac{du}{dy} + \frac{du}{dz}\frac{dz}{dy} = 0,$$

relatively to each of the variables x, y, z.

274. Whenever we have two equations each involving the same three variables, it is obvious that the variables must cease

to be independent of one another, and it will not be difficult to find the relations between the differentials of any two of them considered as functions of the third.

Let $F(x, y, z) = 0$ and $f(x, y, z) = 0$ be the two equations which we will here represent by $u=0$ and $v=0$: then retaining the former notation, we shall have

$$u' = u + \frac{du}{dx}h + \frac{du}{dy}k + \frac{du}{dz}l + \&c.$$

$$+ \&c. \dots\dots\dots\dots\dots\dots$$

$$v' = v + \frac{dv}{dx}h + \frac{dv}{dy}k + \frac{dv}{dz}l + \&c.$$

$$+ \&c. \dots\dots\dots\dots\dots\dots :$$

but, by means of the two proposed equations the quantities x, y may, by elimination, be expressed in terms of z, and therefore we shall have

$$h = \frac{dx}{dz}l + \frac{d^2x}{dz^2}\frac{l^2}{1.2} + \&c.$$

$$k = \frac{dy}{dz}l + \frac{d^2y}{dz^2}\frac{l^2}{1.2} + \&c.:$$

whence we now obtain

$$u' = u + \frac{du}{dx}\left(\frac{dx}{dz}l + \frac{d^2x}{dz^2}\frac{l^2}{1.2} + \&c.\right)$$

$$+ \frac{du}{dy}\left(\frac{dy}{dz}l + \frac{d^2y}{dz^2}\frac{l^2}{1.2} + \&c.\right)$$

$$+ \frac{du}{dz}l + \&c.:$$

and v' may be expressed in a similar form: hence, if by virtue of the equation $u=0$, we equate to zero the sum of the quantities involving the first power of the indeterminate magnitude l, we shall have the results

$$0=\frac{du}{dz}+\frac{du}{dx}\frac{dx}{dz}+\frac{du}{dy}\frac{dy}{dz}=\frac{d(u)}{dz},$$

$$0=\frac{dv}{dz}+\frac{dv}{dx}\frac{dx}{dz}+\frac{dv}{dy}\frac{dy}{dz}=\frac{d(v)}{dz}:$$

and similarly for higher orders.

275. What has just been shewn, proves that the same principles may be extended to any number of equations less by unity than the number of variables involved in them: and whenever this is not the case, we know that there are employed more than one independent principal variable.

Let $u=F(s, t, x, y, z)=0$ and $v=f(s, t, x, y, z)=0$, be two equations between the five variables s, t, x, y, z: then it is obvious that y and z may here be considered as functions of the other variables s, t, x; and differentiating u and v with respect to s, t and x in succession, we shall have by (274),

$$\frac{d(u)}{ds}=\frac{du}{ds}+\frac{du}{dy}\frac{dy}{ds}+\frac{du}{dz}\frac{dz}{ds}=0,$$

$$\frac{d(u)}{dt}=\frac{du}{dt}+\frac{du}{dy}\frac{dy}{dt}+\frac{du}{dz}\frac{dz}{dt}=0,$$

$$\frac{d(u)}{dx}=\frac{du}{dx}+\frac{du}{dy}\frac{dy}{dx}+\frac{du}{dz}\frac{dz}{dx}=0:$$

multiplying these results by ds, dt, dx respectively, adding together the results and putting dy and dz for

$$\frac{dy}{ds}ds+\frac{dy}{dt}dt+\frac{dy}{dx}dx \text{ and } \frac{dz}{ds}ds+\frac{dz}{dt}dt+\frac{dz}{dx}dx,$$

we obtain $du=\frac{du}{ds}ds+\frac{du}{dt}dt+\frac{du}{dx}dx+\frac{du}{dy}dy+\frac{du}{dz}dz=0$: which is the same result as obtainable by the principle of (271) when all the variables are considered independent: and it follows that in differentiating $u=0$ and $v=0$, with respect to all the variables s, t, x, y, z, we may make the coeffi-

cient of the differential of each independent variable separately equal to zero: and similar considerations will be applicable to the investigation of differentials of higher orders.

V. ON THE ELIMINATION OF CONSTANT QUANTITIES AND ARBITRARY FUNCTIONS WHOSE FORMS ARE INDETERMINATE.

276. In articles (67) and (68) it has been shewn that by means of the primitive equation and those successively derived from it by differentiation the constants involved may be made to disappear whenever only one independent variable is concerned; so here if $u=f(x, y, z)=0$, we shall have two differential equations of the first order according as one or other of the independent variables is referred to, and by means of these and the original, we can always eliminate two constant quantities leaving a result existing between x, y, z, $\frac{dz}{dx}$ and $\frac{dz}{dy}$ independent of them: again, by proceeding to the differentials of the second order, we shall manifestly have six equations by means of which may be eliminated five constant magnitudes, and so on: and the same kind of principles will also enable us to exterminate functions of the variables employed whose forms are absolutely unknown.

To exemplify this, let us take the equation $u=F(x, y, z)=0$, which may be supposed to give $z=f(ax+by)$: then if $ax+by=t$, we shall have

$$z=f(t):$$

whence $dz=f'(t)\,dt$ where $f'(t)$ is the differential coefficient $\frac{df(t)}{dt}$ of the function $f(t)$:

$$\therefore \frac{dz}{dx}=f'(t)\frac{dt}{dx} \text{ and } \frac{dz}{dy}=f'(t)\frac{dt}{dy}:$$

but since $t = ax + by$, we have

$$\frac{dt}{dx} = a \text{ and } \frac{dt}{dy} = b,$$

$$\text{so that } \frac{dz}{dx} = af'(t) \text{ and } \frac{dz}{dy} = bf'(t):$$

therefore from the first of these $b\frac{dz}{dx} = abf'(t)$, and from the second $a\frac{dz}{dy} = abf'(t)$: whence $b\frac{dz}{dx} - a\frac{dz}{dy} = 0$: and this is an equation entirely divested of the arbitrary or indeterminate function $f(ax+by)$, but such as holds good whether $f(ax+by)$ represent $(ax+by)^m$, e^{ax+by}, $\log(ax+by)$, $\sin(ax+by)$, &c. as may easily be verified: and conversely, if any expression be such as to satisfy the equation $b\frac{dz}{dx} - a\frac{dz}{dy} = 0$, we may rest assured that it is of the form $z = f(ax+by)$.

277. Generally if we have $u = F\{x, y, z, f(t)\} = 0$, where t is a function of x, y, z, it is evident that two of these quantities z and $f(t)$ may be considered to be functions of the two others x, y: whence we shall have

$$u = 0,$$

$$\frac{du}{dx} + \frac{du}{dz}\frac{dz}{dx} + \frac{du}{df(t)}f'(t)\frac{dt}{dx} = 0,$$

$$\frac{du}{dy} + \frac{du}{dz}\frac{dz}{dy} + \frac{du}{df(t)}f'(t)\frac{dt}{dy} = 0;$$

by means of which a final equation may be derived entirely independent of both $f(t)$ and $f'(t)$.

Ex. 1. Let us take $u = F(x, y, z) = 0$, such that

$$z = f(x^2 + y^2):$$

then $\frac{dz}{dx}=f'(x^2+y^2)\,2x$, $\frac{dz}{dy}=f'(x^2+y^2)\,2y$, also $t=x^2+y^2$: whence we have

$$\frac{dt}{dx}=2x, \text{ and } \frac{dt}{dy}=2y;$$

$$\therefore \frac{du}{dx}+\frac{du}{dz}f'(x^2+y^2)\,2x+\frac{du}{df(t)}f'(t)\,2x=0,$$

$$\frac{du}{dy}+\frac{du}{dz}f'(x^2+y^2)\,2y+\frac{du}{df(t)}f'(t)\,2y=0;$$

wherefore multiplying the former of these by y and the latter by x, and subtracting, we obtain

$$y\frac{du}{dx}-x\frac{du}{dy}=0:$$

but by virtue of the equation $u=0$, it is manifest that

$$\frac{du}{dx}=\frac{dz}{dx} \text{ and } \frac{du}{dy}=\frac{dz}{dy},$$

and therefore we have

$$y\frac{dz}{dx}-x\frac{dz}{dy}=0,$$

which exhibits no trace of the function $f(x^2+y^2)$.

Here we have shewn how the preceding principles are to be applied: but in fact, since

$$\frac{dz}{dx}=f'(x^2+y^2)\,2x \text{ and } \frac{dz}{dy}=f'(x^2+y^2)\,2y,$$

if the first be multiplied by y and the second by x, we obtain immediately the same result as before,

$$y\frac{dz}{dx}-x\frac{dz}{dy}=0.$$

Hence conversely, by means of this partial differential equation

$$y\frac{dz}{dx} - x\frac{dz}{dy} = 0,$$

we shall always be enabled to recognize z as a function of $x^2 + y^2$.

Ex. 2. Let $z = y^2 + 2\phi\left(\frac{1}{x} + \log y\right)$:

$$\text{then } \frac{dz}{dx} = -\frac{2}{x^2}\phi'\left(\frac{1}{x} + \log y\right),$$

$$\frac{dz}{dy} = 2y + \frac{2}{y}\phi'\left(\frac{1}{x} + \log y\right):$$

whence multiplying the former by x^2 and the latter by y, and adding together the two results, we obtain

$$x^2\frac{dz}{dx} + y\frac{dz}{dy} - 2y^2 = 0.$$

278. The same principles may be extended to the elimination of indeterminate functions of more than two independent variables, as may easily be demonstrated generally, and is evinced in the following example.

Ex. To eliminate the arbitrary function from the equation

$$z = xf\left(\frac{y}{x}, \frac{x}{t}\right),$$

we have

$$\frac{dz}{dx} = f\left(\frac{y}{x}, \frac{x}{t}\right) - \frac{y}{x}f'\left(\frac{y}{x}, \frac{x}{t}\right) + \frac{x}{t}f'\left(\frac{y}{x}, \frac{x}{t}\right),$$

$$\frac{dz}{dy} = f'\left(\frac{y}{x}, \frac{x}{t}\right), \quad \frac{dz}{dt} = -\frac{x^2}{t^2}f'\left(\frac{y}{x}, \frac{x}{t}\right):$$

whence we obtain $$\frac{dz}{dx} = \frac{z}{x} - \frac{y}{x}\frac{dz}{dy} - \frac{t}{x}\frac{dz}{dt},$$

which gives immediately

$$\frac{xdz}{dx}+\frac{ydz}{dy}+\frac{tdz}{dt}-z=0.$$

279. By proceeding to higher orders of partial differential coefficients, it will be possible to eliminate in the same manner two or more indeterminate functions from an equation.

Ex. Let $z=xf\left(\frac{y}{x}\right)+\phi\,(yx)$: then putting

$$\frac{dz}{dx}=p,\ \frac{dz}{dy}=q:\ \frac{dp}{dx}=r,\ \frac{dp}{dy}=\frac{dq}{dx}=s,\ \frac{dq}{dy}=t,$$

and taking the partial differential coefficients, we have

$$z=xf\left(\frac{y}{x}\right)+\phi\,(yx),$$

$$p=f\left(\frac{y}{x}\right)-\frac{y}{x}f'\left(\frac{y}{x}\right)+y\phi'\,(yx),$$

$$q=f'\left(\frac{y}{x}\right)+x\phi'\,(yx):$$

whence by multiplication and addition we get

$$px+qy=xf\left(\frac{y}{x}\right)+2xy\phi'\,(yx),$$

from which subtracting the proposed equation, we have

$$px+qy-z=2xy\phi'\,(yx)-\phi\,(yx):$$

differentiating this equation with respect to x and y in succession, we obtain

$$rx+sy=y\phi'\,(yx)+2xy^2\phi''\,(yx),$$

$$sx+ty=x\phi'\,(yx)+2x^2y\phi''\,(yx):$$

which by multiplication and subtraction lead immediately to

$$rx^2 - ty^2 = 0,$$

$$\text{or } x^2 \frac{d^2 z}{dx^2} - y^2 \frac{d^2 z}{dy^2} = 0.$$

280. We have seen that the differential coefficients of the first order are sufficient for the elimination of one indeterminate function, and that those of the second have here been the means of causing the disappearance of two, but we cannot hence infer that three differentiations will enable us to get rid of as many functions: for, to find the order of differentiation to which we must proceed so as to eliminate any number of independent functions from an equation containing two variables, suppose $u = 0$ to comprise m such functions as $f(x, y)$, $\phi(x, y)$, $\psi(x, y)$, &c: then it is manifest that by each successive differentiation we introduce m independent functions, so that by proceeding to the n^{th} order of differentiation, we shall have $(n+1)\,m$ such functions including the original: but $u = 0$ gives $\dfrac{du}{dx} = 0$ and $\dfrac{du}{dy} = 0$ of the first order; $\dfrac{d^2u}{dx^2} = 0$, $\dfrac{d^2u}{dx\,dy} = 0$ and $\dfrac{d^2u}{dy^2} = 0$ of the second order: similarly we have seen that there will be obtained 4, 5, &c. $(n+1)$ equations of the third, fourth, &c. n^{th} orders: whence the entire number of equations altogether

$$= 1 + 2 + 3 + \&\text{c.} + (n+1) = \frac{(n+1)(n+2)}{1\,.\,2}:$$

therefore in order that we may be enabled to eliminate all the proposed functions, we must manifestly have

$$\frac{(n+1)(n+2)}{1\,.\,2} > (n+1)\,m,$$

and therefore $n + 2 > 2m$ or $n > 2m - 2$: that is, n which expresses the order of differentiation must be equal to $2m - 1$ at least; and the number of equations resulting from the elimination being generally

$$\frac{(n+1)(n+2)}{1.2} - (n+1)m = \frac{n+1}{1.2}\{n+2-2m\},$$

it follows that when $n=2m-1$, the said number will be $\frac{1}{2}(n+1)=m$.

Thus, in (276) we have one indeterminate function $f(ax+by)$ and one resulting partial differential equation

$$b\frac{dz}{dx} - a\frac{dz}{dy} = 0:$$

and in the instance of (279) where $m=2$, we should have $n=2.2-1=3$ at least; but from the peculiar form of the functions involved, it is here unnecessary to proceed further than two differentiations, and there may result from it a second partial differential equation in addition to the one there given, namely

$$x^2\frac{d^2z}{dx^2} + 2xy\frac{d^2z}{dx\,dy} + y^2\frac{d^2z}{dy^2} = 0.$$

VI. ON THE DEVELOPEMENT OF FUNCTIONS BY MEANS OF LAGRANGE's THEOREM. ON LAPLACE's THEOREM.

281. If $y=z+x\phi(y)$ where x and z are independent of each other, and $u=f(y)$, then may the developement of u be obtained from the following theorem:

$$u=f(z)+\phi(z)\frac{df(z)}{dz}\frac{x}{1}+\frac{d}{dz}\left\{\overline{\phi(z)}|^2\frac{df(z)}{dz}\right\}\frac{x^2}{1.2}$$

$$+\frac{d^2}{dz^2}\left\{\overline{\phi(z)}|^3\frac{df(z)}{dz}\right\}\frac{x^3}{1.2.3}+\&c.$$

$$+\frac{d^{n-1}}{dz^{n-1}}\left\{\overline{\phi(z)}|^n\frac{df(z)}{dz}\right\}\frac{x^n}{1.2.3.\&c.\,n}+\&c.$$

where $\frac{d}{dz}$, $\frac{d^2}{dz^2}$, &c. $\frac{d^{n-1}}{dz^{n-1}}$ denote the successive differential coefficients of the expressions within the brackets relatively to z.

For, by *Maclaurin's* Theorem, we have

$$u = U_0 + U_1 \frac{x}{1} + U_2 \frac{x^2}{1.2} + U_3 \frac{x^3}{1.2.3} + \&c.$$

$$+ U_n \frac{x^n}{1.2.3.\&c.\,n} + \&c.$$

where U_0, U_1, U_2, U_3, &c., U_n are the values of

$$u, \quad \frac{du}{dx}, \quad \frac{d^2u}{dx^2}, \quad \frac{d^3u}{dx^3}, \quad \&c. \quad \frac{d^nu}{dx^n},$$

when $x = 0$, which it is therefore required to find:

now since $u = f(y)$ and y is a function of x and z, we have

$$\frac{du}{dx} = \frac{df(y)}{dy}\frac{dy}{dx} \text{ and } \frac{du}{dz} = \frac{df(y)}{dy}\frac{dy}{dz}:$$

$$\text{whence } \frac{du}{dx}\frac{dy}{dz} = \frac{du}{dz}\frac{dy}{dx} \dots\dots\dots\dots\dots (1):$$

again, from $y = z + x\phi(y)$, we have

$$\frac{dy}{dx} = \phi(y) + \frac{xd\phi(y)}{dy}\frac{dy}{dx},$$

$$\frac{dy}{dz} = 1 + \frac{xd\phi(y)}{dy}\frac{dy}{dz}:$$

$$\text{whence } \frac{dy}{dx} = \frac{\phi(y)}{1 - \frac{xd\phi(y)}{dy}},$$

$$\text{and therefore } \frac{dy}{dz} = \frac{1}{1 - \frac{xd\phi(y)}{dy}} = \frac{1}{\phi(y)}\frac{dy}{dx};$$

$$\therefore\ \frac{dy}{dx} = \phi(y)\frac{dy}{dz}:$$

$$\therefore\ \phi(y)\frac{dy}{dz}\frac{du}{dz} = \frac{dy}{dx}\frac{du}{dz} = \frac{du}{dx}\frac{dy}{dz} \text{ from (1)}:$$

$$\text{whence } \frac{du}{dx} = \phi(y)\frac{du}{dz} \ldots\ldots\ldots\ldots\ldots (2):$$

again, if $\phi(y)\dfrac{du}{dz} = \dfrac{du_1}{dz}$, u_1 being some function of y, we have

$$\frac{d^2u}{dx^2} = \frac{d^2u_1}{dz\,dx}:$$

$$\text{but by (2), } \frac{du_1}{dx} = \phi(y)\frac{du_1}{dz} = \overline{\phi(y)}|^2\frac{du}{dz};$$

$$\therefore\ \frac{d^2u_1}{dz\,dx} = \frac{d}{dz}\left\{\overline{\phi(y)}|^2\frac{du}{dz}\right\}:$$

$$\text{that is, } \frac{d^2u}{dx^2} = \frac{d}{dz}\left\{\overline{\phi(y)}|^2\frac{du}{dz}\right\}:$$

also, if $\overline{\phi(y)}|^2\dfrac{du}{dz} = \dfrac{du_2}{dz}$, where u_2 is some function of y, we shall readily obtain

$$\frac{d^2u}{dx^2} = \frac{d^2u_2}{dz^2} \text{ and } \therefore\ \frac{d^3u}{dx^3} = \frac{d^3u_2}{dz^2\,dx}:$$

$$\text{but by (2), } \frac{du_2}{dx} = \phi(y)\frac{du_2}{dz} = \overline{\phi(y)}|^3\frac{du}{dz};$$

$$\therefore\ \frac{d^3u}{dx^3} = \frac{d^2}{dz^2}\left\{\overline{\phi(y)}|^3\frac{du}{dz}\right\}: \text{ and so on}:$$

and generally if for the $(n-1)^{\text{th}}$ differential coefficient, we have

$$\frac{d^{n-1}u}{dx^{n-1}} = \frac{d^{n-2}}{dz^{n-2}}\left\{\overline{\phi(y)}|^{n-1}\frac{du}{dz}\right\},$$

and for $\overline{\phi(y)}|^{n-1}\dfrac{du}{dz}$, put $\dfrac{du_{n-1}}{dz}$, then will

$$\frac{d^{n-1}u}{dx^{n-1}} = \frac{d^{n-1}u_{n-1}}{dz^{n-1}} \text{ and } \therefore \frac{d^n u}{dx^n} = \frac{d^n u_{n-1}}{dz^{n-1}dx};$$

$$\text{but by (2), } \frac{du_{n-1}}{dx} = \phi(y)\frac{du_{n-1}}{dz} = \overline{\phi(y)}|^n \frac{du}{dz},$$

$$\therefore \frac{d^n u}{dx^n} = \frac{d^{n-1}}{dz^{n-1}}\left\{\overline{\phi(y)}|^n \frac{du}{dz}\right\}:$$

whence it follows that if this form be true for the differential coefficient of any one order, it will manifestly be true for the next superior order: but it has been shewn to hold good where n is taken successively equal to 1, 2, 3, and therefore it is generally true:

and to find its value when $x=0$, we observe that then $y=z$, and $u=f(y)=f(z)$, so that $U_n = \dfrac{d^{n-1}}{dz^{n-1}}\left\{\overline{\phi(z)}|^n \dfrac{df(z)}{dz}\right\}$: consequently by substituting in the formula of *Maclaurin* above cited, the several values of U_0, U_1, U_2, &c. U_n, &c. we have

$$u = f(z) + \phi(z)\frac{df(z)}{dz}\frac{x}{1} + \frac{d}{dz}\left\{\overline{\phi(z)}|^2 \frac{df(z)}{dz}\right\}\frac{x^2}{1.2}$$

$$+ \frac{d^2}{dz^2}\left\{\overline{\phi(z)}|^3 \frac{df(z)}{dz}\right\}\frac{x^3}{1.2.3} + \&c.$$

$$+ \frac{d^{n-1}}{dz^{n-1}}\left\{\overline{\phi(z)}|^n \frac{df(z)}{dz}\right\}\frac{x^n}{1.2.3.\&c.\,n} + \&c.$$

which is the celebrated Theorem of *Lagrange*, who expressed it originally in the form

$$u = f(z) + \frac{x}{1}f'(z)\phi z + \frac{x^2}{1.2}\{f'(z)\phi z^2\}'$$

$$+ \frac{x^3}{1.2.3}(f' z\phi z^3)'' + \&c.:$$

and its extensive utility will be evinced in the following examples.

Ex. 1. Let $1-y+ay=0$ be given, to find the value of y^m.

The theorem has $y=z+x\phi(y)$ or $z-y+x\phi(y)=0$, which, by putting for z, y and $x\phi(y)$ the quantities 1, y and ay respectively, coincides with the proposed expression: also in this case we have $u=f(y)=y^m$: whence we see that $f(z)=z^m$: and $x\phi(y)$ is here equivalent to ay, so that

$$x=a,\ \phi(y)=y \text{ and } \therefore \phi(z)=z:$$

and we are now enabled to find the several terms as follows:

$$f(z)=z^m=1:$$

$$\phi(z)\frac{df(z)}{dz}\frac{x}{1}=\frac{zd(z^m)}{dz}\frac{a}{1}=mz^m\frac{a}{1}=ma:$$

$$\frac{d}{dz}\left\{\overline{\phi(z)}|^2\frac{df(z)}{dz}\right\}\frac{x^2}{1.2}=\frac{d}{dz}\left\{\frac{z^2d(z^m)}{dz}\right\}\frac{a^2}{1.2}$$

$$=\frac{d}{dz}\left\{mz^{m+1}\right\}\frac{a^2}{1.2}=m(m+1)z^m\frac{a^2}{1.2}$$

$$=\frac{m(m+1)}{1.2}a^2:$$

$$\frac{d^2}{dz^2}\left\{\overline{\phi(z)}|^3\frac{df(z)}{dz}\right\}\frac{x^3}{1.2.3}=\frac{d^2}{dz^2}\left\{\frac{z^3d(z^m)}{dz}\right\}\frac{a^3}{1.2.3}$$

$$=\frac{d^2}{dz^2}\left\{mz^{m+2}\right\}\frac{a^3}{1.2.3}=\frac{d}{dz}\left\{m(m+2)z^{m+1}\right\}\frac{a^3}{1.2.3}$$

$$=\frac{m(m+1)(m+2)}{1.2.3}a^3: \text{ and so on;}$$

whence we obtain by substitution in the theorem

$$y^m=1+ma+\frac{m(m+1)}{1.2}a^2+\frac{m(m+1)(m+2)}{1.2.3}a^3+\&c:$$

and we are assured of the correctness of the result since

$$y^m = \frac{1}{(1-a)^m} = (1-a)^{-m},$$

as appears from the solution of the proposed equation.

Ex. 2. Let $a - by + cy^2 = 0$ be given, to find an expression for y.

Here we have $\frac{a}{b} - y + \frac{c}{b}y^2 = 0$, which will coincide with $z - y + x\phi(y) = 0$ by making $z = \frac{a}{b}$, $x = \frac{c}{b}$ and $\phi(y) = y^2$: whence we have $\phi(z) = z^2$, and $f(z) = z$ since $u = f(y) = y$: therefore the terms in order will be as under:

$$f(z) = z = \frac{a}{b}:$$

$$\phi(z)\frac{df(z)}{dz}\frac{x}{1} = \frac{z^2 dz}{dz}\frac{c}{b} = z^2\frac{c}{b} = \frac{a^2c}{b^3}:$$

$$\frac{d}{dz}\left\{\overline{\phi(z)}|^2\frac{df(z)}{dz}\right\}\frac{x^2}{1.2} = \frac{d}{dz}\left\{z^4\frac{dz}{dz}\right\}\frac{c^2}{1.2b^2}$$

$$= 4z^3\frac{c^2}{1.2b^2} = 4\frac{a^3c^2}{1.2b^5}:$$

$$\frac{d^2}{dz^2}\left\{\overline{\phi(z)}|^3\frac{df(z)}{dz}\right\}\frac{x^3}{1.2.3} = \frac{d^2}{dz^2}\left\{z^6\frac{dz}{dz}\right\}\frac{c^3}{1.2.3b^3}$$

$$= 5.6z^4\frac{c^3}{1.2.3b^3} = 5.6\frac{a^4c^3}{1.2.3b^7};$$

$$\frac{d^3}{dz^3}\left\{\overline{\phi(z)}|^4\frac{df(z)}{dz}\right\}\frac{x^4}{1.2.3.4} = \frac{d^3}{dz^3}\left\{z^8\frac{dz}{dz}\right\}\frac{c^4}{1.2.3.4b^4}$$

$$= 6.7.8z^5\frac{c^4}{1.2.3.4b^4} = 6.7.8\frac{a^5c^4}{1.2.3.4b^9}; \text{ and so on;}$$

whence we have

$$y = \frac{a}{b} + \frac{a^2 c}{b^3} + 4\frac{a^3 c^2}{1.2 b^5} + 5.6\frac{a^4 c^3}{1.2.3 b^7} + 6.7.8\frac{a^5 c^4}{1.2.3.4 b^9} + \&c.$$

which, it may be remarked, is the same as $\dfrac{b-\sqrt{b^2-4ac}}{2c}$ the less root of the equation solved by the ordinary means.

Ex. 3. Given $xy^n - y + a = 0$, to find the value of y^m.

Here comparing $a - y + xy^n$ with $z - y + x\phi(y)$, we have $z = a$ and $\phi(y) = y^n$: also $u = f(y) = y^m$; and $\therefore\ \phi(z) = z^n$, $f(z) = z^m$: whence we obtain the terms immediately:

$$f(z) = z^m = a^m;$$

$$\phi(z)\frac{df(z)}{dz}\frac{x}{1} = z^n \frac{d(z^m)}{dz}\frac{x}{1} = mz^{m+n-1}x$$

$$= ma^{m+n-1}x:$$

$$\frac{d}{dz}\left\{\overline{\phi(z)}|^2 \frac{df(z)}{dz}\right\}\frac{x^2}{1.2} = \frac{d}{dz}\left\{z^{2n}\frac{d(z^m)}{dz}\right\}\frac{x^2}{1.2}$$

$$= \frac{d}{dz}\left\{mz^{m+2n-1}\right\}\frac{x^2}{1.2} = m(m+2n-1)\,z^{m+2n-2}\frac{x^2}{1.2}$$

$$= \frac{m(m+2n-1)}{1.2}a^{m+2n-2}x^2:$$

$$\frac{d^2}{dz^2}\left\{\overline{\phi(z)}|^3 \frac{df(z)}{dz}\right\}\frac{x^3}{1.2.3} = \frac{d^2}{dz^2}\left\{z^{3n}\frac{d(z^m)}{dz}\right\}\frac{x^3}{1.2.3}$$

$$= \frac{d^2}{dz^2}\left\{mz^{m+3n-1}\right\}\frac{x^3}{1.2.3} =$$

$$m(m+3n-1)(m+3n-2)\,z^{m+3n-3}\frac{x^3}{1.2.3}$$

$$= \frac{m(m+3n-1)(m+3n-2)}{1.2.3}a^{m+3n-3}x^3: \text{ and so on:}$$

$$\therefore\ y^m = a^m + m a^{m+n-1} x + \frac{m(m+2n-1)}{1.2} a^{m+2n-2} x^2$$

$$+ \frac{m(m+3n-1)(m+3n-2)}{1.2.3} a^{m+3n-3} x^3 + \&c.$$

If $m=1$ and n be taken equal to 2, 3, &c. in succession, we shall have the least root of the corresponding quadratic, cubic, &c. equation.

Ex. 4. Given $y^3 - 2xy^2 + x^2y - a^3 = 0$, to find the values of y.

Here we have $x - y \pm \dfrac{a^{\frac{3}{2}}}{y^{\frac{1}{2}}} = 0$ by solving the equation with respect to x: and comparing this with $z - y + x\phi(y) = 0$, we find $z = x$, $x = \pm a^{\frac{3}{2}}$ and $\phi(y) = y^{-\frac{1}{2}}$; also $u = f(y) = y$: whence $\phi(z) = z^{-\frac{1}{2}}$, $f(z) = z$ and $\dfrac{df(z)}{dz} = 1$: therefore the terms are as follow:

$$f(z) = z = x:$$

$$\phi(z)\frac{df(z)}{dz}\frac{x}{1} = \pm z^{-\frac{1}{2}} a^{\frac{3}{2}} = \pm \frac{a^{\frac{3}{2}}}{x^{\frac{1}{2}}}:$$

$$\frac{d}{dz}\left\{\overline{\phi(z)}|^2 \frac{df(z)}{dz}\right\}\frac{x^2}{1.2} = \frac{d}{dz}\{z^{-1}\}\frac{a^3}{1.2} = -z^{-2}\frac{a^3}{1.2}$$

$$= -\frac{a^3}{1.2x^2}:$$

$$\frac{d^2}{dz^2}\left\{\overline{\phi(z)}|^3 \frac{df(z)}{dz}\right\}\frac{x^3}{1.2.3} = \frac{d^2}{dz}\{z^{-\frac{3}{2}}\}\frac{\pm a^{\frac{9}{2}}}{1.2.3}$$

$$= \pm \frac{3.5}{2^2} z^{-\frac{7}{2}} \frac{a^{\frac{9}{2}}}{1.2.3} = \pm \frac{3.5a^{\frac{9}{2}}}{2^2.1.2.3x^{\frac{7}{2}}}: \text{ and so on:}$$

$$\therefore\ y = x \pm \frac{a^{\frac{3}{2}}}{x^{\frac{1}{2}}} - \frac{a^3}{1.2x^2} \pm \frac{3.5a^{\frac{9}{2}}}{2^2.1.2.3x^{\frac{7}{2}}} - \&c.$$

Also, since $y = \frac{a^3}{x^2} + \left(\frac{2y^2}{x} - \frac{y^3}{x^2}\right)$, the theorem gives likewise $y = \frac{a^3}{x^2} + \frac{2a^6}{x^5} - \frac{7a^9}{x^8} + \&c.$

Ex. 5. To find the roots of the equation $a - bx + cx^n = 0$, let $y = x^n$, which gives $-\frac{a}{c} - y + \frac{b}{c}y^{\frac{1}{n}} = 0$: and comparing this with $z - y + x\phi(y) = 0$, we have $z = -\frac{a}{c}$, $x = \frac{b}{c}$ and $\phi(y) = y^{\frac{1}{n}}$, also $u = f(y) = y$:

$$\text{therefore } \phi(z) = z^{\frac{1}{n}},\ f(z) = z \text{ and } \frac{df(z)}{dz} = 1:$$

and the terms being found as before, we obtain

$$y = \alpha\left\{1 - \frac{1}{n}\frac{b\alpha}{a} - \frac{(n-2)}{1.2\,n^2}\frac{b^2\alpha^2}{a^2} - \frac{(n-3)(n-4)}{1.2.3n^3}\frac{b^3\alpha^3}{a^3} - \&c.\right\}$$

$$\text{where } \alpha = \left(-\frac{a}{c}\right)^{\frac{1}{n}}:$$

$$\text{but since } -\frac{a}{c} = -\frac{a}{c}\{\cos 2m\pi + \sqrt{-1}\sin 2m\pi\},$$

$$\text{we have } \alpha = \sqrt[n]{-\frac{a}{c}}\left\{\cos\frac{2m\pi}{n} + \sqrt{-1}\sin\frac{2m\pi}{n}\right\};$$

from which, by assigning to m the successive values 1, 2, 3, &c. n, we shall find the n different roots.

Ex. 6. Given $1 - x + e^x = 0$, to find the value of e^x.

Here changing x into y for the sake of coincidence with the general formula $z - y + x\phi(y) = 0$, we have $1 - y + e^y = 0$:

$$\therefore\ z = 1,\ x = 1,\ \phi(y) = e^y,\ u = f(y) = e^y:$$

$$\therefore\ \phi(z) = e^z,\ f(z) = e^z \text{ and } \frac{df(z)}{dz} = e^z:$$

whence we may find the terms as before:

$$f(z) = e^z = e:$$

$$\phi(z) \frac{df(z)}{dz} \frac{x}{1} = e^{2z} = e^2:$$

$$\frac{d}{dz}\left\{\overline{\phi(z)}|^2 \frac{df(z)}{dz}\right\} \frac{x^2}{1.2} = \frac{d}{dz}\{e^{3z}\} \frac{1}{1.2} = 3e^{3z}\frac{1}{1.2} = \frac{3e^3}{1.2}:$$

$$\frac{d^2}{dz^2}\left\{\overline{\phi(z)}|^3 \frac{df(z)}{dz}\right\} \frac{x^3}{1.2.3} =$$

$$\frac{d^2}{dz^2}\{e^{4z}\}\frac{1}{1.2.3} = 4^2 e^{4z}\frac{1}{1.2.3} = \frac{4^2 e^4}{1.2.3};$$

$$\frac{d^3}{dz^3}\left\{\overline{\phi(z)}|^4 \frac{df(z)}{dz}\right\} \frac{x^4}{1.2.3.4} =$$

$$\frac{d^3}{dz^3}\{e^{5z}\}\frac{1}{1.2.3.4} = 5^3 e^{5z}\frac{1}{1.2.3.4} = \frac{5^3 e^5}{1.2.3.4};$$ and so on:

whence we have

$$e^y = e + e^2 + \frac{3e^3}{1.2} + \frac{4^2 e^4}{1.2.3} + \frac{5^3 e^5}{1.2.3.4} + \&c.$$

which by putting x for y becomes

$$e^x = e\left\{1 + \frac{2e}{1.2} + \frac{3^2 e^2}{1.2.3} + \frac{4^3 e^3}{1.2.3.4} + \frac{5^4 e^4}{1.2.3.4.5} + \&c.\right\}$$

in which the law is manifest.

Ex. 7. Given $x^2 - 2x + 1 = 0$, to find $\log x$.

Here changing the letter as before, we have $\frac{1}{2} - y + \frac{1}{2}y^2 = 0$ to be compared with $z - y + x\phi(y) = 0$: whence $z = \frac{1}{2}$, $x = \frac{1}{2}$, $\phi(y) = y^2$ and $u = f(y) = \log y$:

$$\therefore \phi(z) = z^2,\ f(z) = \log z \text{ and } \frac{df(z)}{dz} = \frac{1}{z}:$$

whence $f(z) = \log z = \log \frac{1}{2}$:

$$\phi(z) \frac{df(z)}{dz} \frac{x}{1} = z\frac{1}{2} = \frac{1}{2^2}:$$

$$\frac{d}{dz}\left\{\overline{\phi(z)}|^2 \frac{df(z)}{dz}\right\} \frac{x^2}{1 \cdot 2} = 3z^2 \frac{1}{1 \cdot 2 \cdot 2^2} = \frac{3}{1 \cdot 2 \cdot 2^4}:$$

$$\frac{d^2}{dz^2}\left\{\overline{\phi(z)}|^3 \frac{df(z)}{dz}\right\} \frac{x^3}{1 \cdot 2 \cdot 3} = 4 \cdot 5z^3 \frac{1}{1 \cdot 2 \cdot 3 \cdot 2^3} = \frac{4 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 2^4}: \text{ \&c.}$$

$$\therefore \log y = \log \tfrac{1}{2} + \frac{1}{2^2} + \frac{3}{1 \cdot 2 \cdot 2^4} + \frac{4 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 2^4} + \text{\&c.}$$

which is also the value of $\log x$ from the equation proposed.

Ex. 8. Given $y = \dfrac{e}{1 + \sqrt{1 - e^2}}$, to develope y^n in ascending powers of e.

This quantity being the less root of the equation

$$y^2 - \frac{2}{e} y + 1 = 0,$$

we have to compare $\frac{1}{2} e - y + \frac{1}{2} ey^2 = 0$ with $z - y + x\phi(y) = 0$; and this gives $z = \frac{1}{2} e$, $x = \frac{1}{2} e$, $\phi(y) = y^2$; also $u = f(y) = y$; whence it follows that $\phi(z) = z^2$, $f(z) = z$ and $\dfrac{df(z)}{dz} = 1$; and the values of the several terms being found and substituted in the general formula give

$$y^n = \frac{e^n}{2^n}\left\{1 + n\frac{e^2}{2^2} + \frac{n(n+3)}{1 \cdot 2}\frac{e^4}{2^4} + \frac{n(n+4)(n+5)}{1 \cdot 2 \cdot 3}\frac{e^6}{2^6} + \text{\&c.}\right\}.$$

If $n = 1$, we have

$$y = \frac{e}{2}\left\{1 + \frac{e^2}{2^2} + 2\frac{e^4}{2^4} + 5\frac{e^6}{2^6} + \text{\&c.}\right\}:$$

and if $n = -1$, we obtain from the formula,

$$\frac{1}{y} = \frac{2}{e}\left\{1 - \frac{e^2}{2^2} - \frac{e^4}{2^4} - 2\frac{e^6}{2^6} - \&c.\right\}$$

which is therefore the developement of

$$\frac{1+\sqrt{1-e^2}}{e} \text{ or } \frac{e}{1-\sqrt{1-e^2}}.$$

Ex. 9. Let $u = nt + e \sin u$, which by changing the letter u into y becomes $nt - y + e \sin y = 0$; and this compared with $z - y + x\phi(y) = 0$ gives $z = nt$, $x = e$, $\phi(y) = \sin y$: also $u = f(y) = y$: whence $\phi(z) = \sin z$, $f(z) = z$ and $\dfrac{df(z)}{dz} = 1$: and the proper operations being effected, we have immediately

$$u = nt + e \sin nt + \frac{e^2}{1.2.2^2} 2 \sin 2nt$$

$$+ \frac{e^3}{1.2.3.2^2}(3^2 \sin 3nt - 3 \sin nt) + \&c.$$

Ex. 10. Given $r = \dfrac{a(1-e^2)}{1 + e\cos v}$, to find the developement of r^n.

Here $r + re\cos v = a(1-e^2)$, which, if y be put for r, becomes $a(1-e^2) - y - ey\cos v = 0$:

$\therefore\ z = a(1-e^2)$, $x = -e\cos v$, $\phi(y) = y$: also $u = f(y) = y^n$;

and thence we have

$$\phi(z) = z,\ f(z) = z^n \text{ and } \frac{df(z)}{dz} = nz^{n-1}:$$

and the theorem immediately gives

$$r^n = a^n(1-e^2)^n\left\{1 - \frac{n}{1} e\cos v + \frac{n(n+1)}{1.2} e^2 \cos^2 v - \&c.\right\}.$$

If $n=1$, we shall have

$$r=a(1-e^2)\{1-e\cos v+e^2\cos^2 v-\&c.\},$$

as may be found by actual division.

Ex. 11. Given $ay+by^2+cy^3+\&c.=\alpha x+\beta x^2+\gamma x^3+\&c.$, to exhibit y in terms of x.

Here $y=\dfrac{1}{a}(\alpha x+\beta x^2+\gamma x^3+\&c.)-\dfrac{1}{a}(by^2+cy^3+\&c.)$, which being compared with the general form, gives

$$z=\frac{1}{a}(\alpha x+\beta x^2+\gamma x^3+\&c.),$$

$$x=-\frac{1}{a} \text{ and } \phi(y)=by^2+cy^3+\&c.: \text{ also } u=f(y)=y:$$

whence we have

$$\phi(z)=bz^2+cz^3+\&c.,\ f(z)=z \text{ and } \frac{df(z)}{dz}=1;$$

and effecting the proper operations, we find

$$y=\frac{1}{a}\{\alpha x+\beta x^2+\gamma x^3+\&c.\}$$

$$-\frac{1}{a}\left\{\frac{b}{a^2}(\alpha x+\beta x^2+\&c.)^2+\frac{c}{a^3}(\alpha x+\beta x^2+\&c.)^3\right\}$$

$$+\frac{2}{a^2}\left\{\frac{b^2}{a^4}(\alpha x+\beta x^2+\&c.)^3(\alpha+2\beta x+\&c.)+\&c.\right\}$$

$$-\&c.\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$=\frac{\alpha}{a}x+\left(\frac{\beta}{a}-\frac{\alpha^2 b}{a^3}\right)x^2+\&c.$$

If $\beta=\gamma=\&c.=0$, the equation proposed becomes

$$ay+by^2+cy^3+\&c.=\alpha x,$$

and we have

$$y = \frac{a}{a}x - \frac{a^2 b}{a^3}x^2 + \frac{2a^3b^2 - a^3ac}{a^5}x^3 - \&c.$$

Ex. 12. Given $u + \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\frac{h^3}{1.2.3} + \&c. = 0$, to find h in terms of u.

Here changing the letter h into y, we shall have

$$-\frac{u}{p} - y - \frac{1}{p}\left(\frac{qy^2}{1.2} + \frac{ry^3}{1.2.3} + \&c.\right) = 0;$$

p, q, r, &c. being taken to represent the successive differential coefficients of u: and this being made to coincide with $z - y + x\phi(y) = 0$ gives

$$z = -\frac{u}{p},\ x = -\frac{1}{p},\ \phi(y) = \frac{qy^2}{1.2} + \frac{ry^3}{1.2.3} + \&c.$$

and $u = f(y) = y$:

$$\therefore\ \phi(z) = \frac{qz^2}{1.2} + \frac{rz^3}{1.2.3} + \&c.,\ f(z) = z \text{ and } \frac{df(z)}{dz} = 1:$$

whence we shall have

$$y = -\left\{\frac{u}{p} + \frac{q}{p^3}\frac{u^2}{1.2} + \frac{3q^2 - pr}{p^5}\frac{u^3}{1.2.3}\right.$$

$$\left. + \frac{3.5q^3 - 2.5pqr + p^2s}{p^7}\frac{u^4}{1.2.3.4} + \&c.\right\},$$

which by restoring the value of y gives the developement required.

282. If $u = f(y)$ and $y = \psi\{z + x\phi(y)\}$, we shall have

$$\frac{dy}{dx} = \psi'\{z + x\phi(y)\}\left\{\phi(y) + x\phi'(y)\frac{dy}{dx}\right\},$$

$$\frac{dy}{dz} = \psi'\{z + x\phi(y)\}\left\{1 + x\phi'(y)\frac{dy}{dz}\right\}:$$

whence to eliminate $\phi'(y)$ we have

$$\frac{dy}{dx}\left\{1 - x\phi'(y)\,\psi'\{z + x\phi(y)\}\right\} = \psi'\{z + x\phi(y)\}\,\phi y,$$

$$\frac{dy}{dz}\left\{1 - x\phi'(y)\,\psi'\{z + x\phi(y)\}\right\} = \psi'\{z + x\phi(y)\},$$

from which it is manifest that there results the equation

$$\frac{dy}{dx} = \phi(y)\frac{dy}{dz},$$

which is likewise independent of the indeterminate function $\psi'\{z + x\phi(y)\}$, and is precisely of the same form as (2) in the demonstration of the Theorem of *Lagrange:* whence proceeding exactly as has there been done, we shall find

$$\frac{d^n u}{dx^n} = \frac{d^{n-1}}{dz^{n-1}}\left\{\overline{\phi(y)}|^n \frac{du}{dz}\right\}:$$

and making $x = 0$ in the values of u and its differential coefficients agreeably to *Maclaurin's* Theorem, we shall have

$$U_0 = f(y) = f\{\psi(z)\},$$

$$U_1 = \phi(y)\frac{df(y)}{dz} = \phi\{\psi(z)\}\,\frac{df\{\psi(z)\}}{dz},$$

$$U_2 = \frac{d}{dz}\left\{\overline{\phi(y)}|^2 \frac{df(y)}{dz}\right\} = \frac{d}{dz}\left\{\overline{\phi\{\psi(z)\}}|^2 \frac{df\{\psi(z)\}}{dz}\right\}:$$

&c...

and consequently by substitution we obtain

$$u = f\{\psi(z)\} + \phi\{\psi(z)\}\,\frac{df\{\psi(z)\}}{dz}\,\frac{x}{1}$$

$$+ \frac{d}{dz}\left\{\overline{\phi\{\psi(z)\}}|^2 \frac{df\{\psi(z)\}}{dz}\right\}\frac{x^2}{1.2} + \text{\&c.};$$

which is the Theorem of *Laplace.*

If $\psi\{z+x\phi(y)\}=z+x\phi(y)$, we may obviously remove the symbol ψ from the equation just found, which then becomes *Lagrange's* Theorem.

Also, if $x=h$ and $\phi(y)=1$, from either of these Theorems we obtain $u=f(y)=f(z+h)=$

$$f(z)+\frac{df(z)}{dz}\frac{h}{1}+\frac{d^2f(z)}{dz^2}\frac{h^2}{1.2}+\frac{d^3f(z)}{dz^3}\frac{h^3}{1.2.3}+\&c.,$$

which is the Theorem of *Taylor*.

CHAP. XIII.

On the Maxima and Minima of Functions of two or more independent Variables.

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283. *To investigate the conditions necessary that* $u=f(x, y)$ *may be a maximum or a minimum, and to find a criterion for deciding whether it is a maximum or a minimum.*

Let h and k as before denote the indeterminate increments of x and y, then we have seen in (259) that

$$\begin{aligned} u' &= f(x+h,\ y+k) \\ &= u + \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \&c. \\ &\qquad + \frac{du}{dy}k + \frac{d^2u}{dx\,dy}kh + \&c. \\ &\qquad\qquad + \frac{d^2u}{dy^2}\frac{k^2}{1.2} + \&c. \\ &\qquad\qquad\qquad + \&c.; \end{aligned}$$

and for the same reason we have

$$\begin{aligned} u_{,} &= f(x-h,\ y-k) \\ &= u - \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{1.2} - \&c. \\ &\qquad - \frac{du}{dy}k + \frac{d^2u}{dx\,dy}kh - \&c. \\ &\qquad\qquad + \frac{d^2u}{dy^2}\frac{k^2}{1.2} - \&c. \\ &\qquad\qquad\qquad - \&c.: \end{aligned}$$

now, if $\frac{du}{dx}$ and $\frac{du}{dy}$ be of finite values, it is obvious that the quantities h and k may be made so small that the first

terms of $u' - u$ and $u_{,} - u$ may be of greater magnitudes than the sums of all those that follow, in which case one of them will be positive and the other negative, and therefore u cannot be a maximum or a minimum: hence, by reason of the independence of x and y, and therefore of h and k, it follows that $\frac{du}{dx} = 0$ and $\frac{du}{dy} = 0$; and these equations will in general be sufficient to ensure to u a maximum or a minimum value:

but when this is the case, it will also be necessary that we have a relation between x and y, such that the second term of $u' - u$ or $u_{,} - u$ may have the same algebraical sign whatever values are given to h and k, whether positive or negative:

let, therefore, $\frac{d^2u}{dx^2} = A$, $\frac{d^2u}{dx\,dy} = B$, $\frac{d^2u}{dy^2} = C$, and this term will become

$$A\frac{h^2}{1.2} + Bhk + C\frac{k^2}{1.2}:$$

which will evidently not change its sign, if it consist of the sum of two squares or multiples of such: now this quantity is obviously equal to

$$\frac{A}{2}\left\{(h + \frac{B}{A}k)^2 + \frac{AC - B^2}{A^2}k^2\right\},$$

and therefore will not change its algebraical sign whatever values are given to h and k, provided AC be not less than B^2: that is, the function admits of a maximum or a minimum whenever A and C have the same algebraical sign, and AC is not less than B^2: and according as A is positive or negative, the corresponding value of u will be a minimum or a maximum, as is evident from this expression: also the three equations, $u = f(x, y)$, $\frac{du}{dx} = 0$ and $\frac{du}{dy} = 0$, will be sufficient for the determination of x, y and u.

Ex. 1. Let $u = x^2 + xy + y^2 - ax - by$:

then $\dfrac{du}{dx} = 2x + y - a = 0,$

and $\dfrac{du}{dy} = x + 2y - b = 0;$

whence $x = \frac{1}{3}(2a - b)$ and $y = \frac{1}{3}(2b - a)$:

now $\dfrac{d^2u}{dx^2} = 2 = A, \quad \dfrac{d^2u}{dx\,dy} = 1 = B$ and $\dfrac{d^2u}{dy^2} = 2 = C;$

that is, $AC = 4$ is greater than $B^2 = 1$:

wherefore since A is positive, these values of x and y render the function a minimum, which is

$$u = -\tfrac{1}{3}(a^2 - ab + b^2).$$

Ex. 2. Let $u = ax^2 + 2bxy + cy^2 - ex - gy$:

then $\dfrac{du}{dx} = 2ax + 2by - e = 0,$

and $\dfrac{du}{dy} = 2bx + 2cy - g = 0$:

from the first $\quad 2acx + 2bcy - ce = 0,$

from the second $2b^2x \;\; + 2bcy - bg = 0$:

$$\therefore\; x = \frac{ce - bg}{2(ac - b^2)} \text{ and } y = \frac{be - ag}{2(ac - b^2)}:$$

also, $\dfrac{d^2u}{dx^2} = 2a = A, \quad \dfrac{d^2u}{dx\,dy} = 2b = B$ and $\dfrac{d^2u}{dy^2} = 2c = C$:

whence it follows that if ac be greater than b^2, we shall have the corresponding value of u a *minimum.*

If a and c be negative and ac be greater than b^2, u will be a *maximum:* but if a and c have different signs, or ac be less than b^2, u will be neither a maximum nor a minimum.

Ex. 3. Given $u = \frac{1}{2}xy + (a - x - y)\left(\frac{x}{3} + \frac{y}{4}\right)$,

$$\text{then } \frac{du}{dx} = \frac{1}{3}a - \frac{2}{3}x - \frac{1}{12}y = 0,$$

$$\text{and } \frac{du}{dy} = \frac{1}{4}a - \frac{1}{12}x - \frac{1}{2}y = 0:$$

from which simultaneously existent equations we obtain

$$x = \frac{21}{47}a \text{ and } y = \frac{20}{47}a:$$

also, $\frac{d^2u}{dx^2} = -\frac{2}{3} = A$, $\frac{d^2u}{dx\,dy} = -\frac{1}{12} = B$, $\frac{d^2u}{dy^2} = -\frac{1}{2} = C$:

that is, A and C have the same algebraical sign, and AC is greater than B^2: wherefore, since A is negative, the proposed function u admits of a maximum value $= \frac{6}{47}a^2$.

Ex. 4. Given $u = \left(1 - \frac{x}{c} - \frac{y}{c}\right)\left(1 - \frac{a}{x} - \frac{b}{y}\right)$:

$$\text{then } \frac{du}{dx} = \frac{a}{x^2}\left(1 - \frac{x}{c} - \frac{y}{c}\right) - \frac{1}{c}\left(1 - \frac{a}{x} - \frac{b}{y}\right) = 0,$$

$$\text{and } \frac{du}{dy} = \frac{b}{y^2}\left(1 - \frac{x}{c} - \frac{y}{c}\right) - \frac{1}{c}\left(1 - \frac{a}{x} - \frac{b}{y}\right) = 0:$$

whence by subtraction we find

$$\left(\frac{a}{x^2} - \frac{b}{y^2}\right)\left(1 - \frac{x}{c} - \frac{y}{c}\right) = 0,$$

which will be satisfied by making each of the factors equal to zero:

$$\text{from the former } y = \pm x\sqrt{\frac{b}{a}}:$$

$$\therefore \frac{a}{x^2}\left(1 - \frac{x}{c} \mp \frac{x}{c}\sqrt{\frac{b}{a}}\right) = \frac{1}{c}\left(1 - \frac{a}{x} \mp \frac{\sqrt{ab}}{x}\right):$$

whence $x = \pm\sqrt{ac}$, and therefore $y = \pm\sqrt{bc}$:

$$\therefore\ u = \left(1 \mp \sqrt{\frac{a}{c}} \mp \sqrt{\frac{b}{c}}\right)\left(1 \mp \sqrt{\frac{a}{c}} \mp \sqrt{\frac{b}{c}}\right)$$

$$= \left(1 \mp \sqrt{\frac{a}{c}} \mp \sqrt{\frac{b}{c}}\right)^2:$$

$$\text{but } \frac{d^2u}{dx^2} = -\frac{2a}{cx^3}(1 + c - x - y) = A,$$

$$\frac{d^2u}{dx\,dy} = -\frac{a}{cx^2}\left(1 + \frac{bx^2}{ay^2}\right) = B,$$

$$\frac{d^2u}{dy^2} = -\frac{2b}{cy^3}(1 + c - x - y) = C,$$

which prove the value of u to be a maximum or a minimum, according as $1 + c - x - y$ is positive or negative, provided AC be greater than B^2:

from the latter we have $x + y = c$,

$$\therefore\ 1 - \frac{a}{x} - \frac{b}{y} = 0, \text{ or } xy - ay - bx = 0:$$

and these two equations give

$$x = \frac{a - b + c \pm \sqrt{(a-b)^2 - 2(a+b)c + c^2}}{2}$$

$$y = \frac{b - a + c \mp \sqrt{(a-b)^2 - 2(a+b)c + c^2}}{2}:$$

$$\text{but } \frac{d^2u}{dx^2} = -\frac{2a}{cx^2} = A,$$

$$\frac{d^2u}{dx\,dy} = -\frac{ab}{cxy}\left(\frac{x}{ay} + \frac{y}{bx}\right) = B,$$

$$\frac{d^2u}{dy^2} = -\frac{2b}{cy^2} = C:$$

which prove that $u=0$ will be a maximum whenever AC is greater than B^2, but neither a maximum nor a minimum when it is less.

Ex. 5. Given $u = x^4 + y^4 - x^2 + xy - y^2$:

$$\text{then } \frac{du}{dx} = 4x^3 - 2x + y = 0,$$

$$\text{and } \frac{du}{dy} = 4y^3 - 2y + x = 0:$$

from which, by the elimination of y, we obtain

$$256x^9 - 384x^7 + 192x^5 - 40x^3 + 3x = 0,$$

which may assume the form

$$x(4x^2 - 1)^3 (4x^2 - 3) = 0,$$

and will therefore be satisfied by making $x = 0$, $4x^2 - 1 = 0$ and $4x^2 - 3 = 0$:

$$\text{also, } \frac{d^2u}{dx^2} = 12x^2 - 2 = A,$$

$$\frac{d^2u}{dx\,dy} = 1 = B,$$

$$\frac{d^2u}{dy^2} = 12y^2 - 2 = C:$$

but, if $x = 0$ and therefore $y = 0$, we have $A = -2$, $B = 1$, $C = -2$, so that $u = 0$ is a *maximum*:

if $x = \pm\frac{1}{2}$, and therefore $y = \pm\frac{1}{2}$, $A = 1$, $B = 1$, $C = 1$, we have $AC = B^2$, whence $u = -\frac{1}{8}$ is a *minimum*:

if $x = \pm\frac{1}{2}\sqrt{3}$, and therefore $y = \mp\frac{1}{2}\sqrt{3}$, $A = 7$, $B = 1$, $C = 7$, we conclude that $u = -\frac{9}{8}$ is a *minimum*.

284. If a proposed function involve three variables subject to an equation of condition, the same process may be used.

For, let $u=f(x, y, z)$ be a maximum or a minimum, subject to the condition expressed by the equation

$$\phi(x, y, z)=a:$$

then, by reason of this equation, any one of the variables z may be regarded as a function of the two others: and consequently from the circumstance of $u=f(x, y, z)$ being a maximum or a minimum, we have

$$P+R\frac{dz}{dx}=0 \text{ and } Q+R\frac{dz}{dy}=0:$$

also from the equation $a=\phi(x, y, z)$ we obtain

$$S+V\frac{dz}{dx}=0 \text{ and } T+V\frac{dz}{dy}=0:$$

whence, eliminating $\frac{dz}{dx}$ and $\frac{dz}{dy}$, we get

$$\frac{P}{R}=\frac{S}{V} \text{ and } \frac{Q}{R}=\frac{T}{V}:$$

which, together with the equation $\phi(x, y, z)=a$, will be sufficient for the determination of the three quantities x, y, z.

Ex. 1. Let $u=x^\alpha y^\beta z^\gamma$ be subject to the condition that $x+y+z=a$: then eliminating z we have

$$u=x^\alpha y^\beta(a-x-y)^\gamma:$$

$$\therefore \frac{du}{dx}=\alpha x^{\alpha-1}y^\beta(a-x-y)^\gamma-\gamma x^\alpha y^\beta(a-x-y)^{\gamma-1},$$

$$\frac{du}{dy}=\beta x^\alpha y^{\beta-1}(a-x-y)^\gamma-\gamma x^\alpha y^\beta(a-x-y)^{\gamma-1}:$$

and these being respectively equated to zero, give

$$\alpha(a-x-y)=\gamma x,$$

$$\beta(a-x-y)=\gamma y:$$

from which we readily find

$$x = \frac{\alpha a}{\alpha+\beta+\gamma}, \; y = \frac{\beta a}{\alpha+\beta+\gamma}, \text{ and thence } z = \frac{\gamma a}{\alpha+\beta+\gamma}:$$

and the usual criterion will shew that

$$u = \frac{\alpha^\alpha \beta^\beta \gamma^\gamma a^{\alpha+\beta+\gamma}}{(\alpha+\beta+\gamma)^{\alpha+\beta+\gamma}} \text{ is a } \textit{maximum}.$$

If $\alpha=\beta=\gamma=1$, we have $x=y=z=\frac{1}{3}a$ and $u=\frac{1}{27}a^3$ a *maximum*.

Ex. 2. Given $u=a^x b^y c^z=$ a maximum or a minimum, to find the values of x, y, z when they are subject to the condition expressed by $(x+1)(y+1)(z+1)=m$.

Taking the logarithms, and considering z as a function of x and y, we have

$$v = \log u = x \log a + y \log b + z \log c;$$

$$\therefore \; \frac{dv}{dx} = \log a + \log c \frac{dz}{dx} = 0,$$

$$\frac{dv}{dy} = \log b + \log c \frac{dz}{dy} = 0:$$

but from the equation of condition

$$\log m = \log(x+1) + \log(y+1) + \log(z+1),$$

we have

$$0 = \frac{1}{x+1} + \frac{1}{z+1}\frac{dz}{dx},$$

$$0 = \frac{1}{y+1} + \frac{1}{z+1}\frac{dz}{dy}:$$

now the former equations give immediately

$$\frac{dz}{dx} = -\frac{\log a}{\log c}, \; \frac{dz}{dy} = -\frac{\log b}{\log c},$$

and from the latter are obtained

$$\frac{dz}{dx} = -\frac{z+1}{x+1}, \quad \frac{dz}{dy} = -\frac{z+1}{y+1}:$$

$$\text{whence } (z+1)\log c = (x+1)\log a,$$

$$\text{and } (z+1)\log c = (y+1)\log b:$$

$$\therefore \frac{(z+1)\log c}{\log a}\,\frac{(z+1)\log c}{\log b}\,(z+1) = m,$$

$$\text{or } (z+1)^3 (\log c)^2 = m \log a \log b,$$

$$\text{and } (z+1)^3 = \frac{m \log a \log b}{(\log c)^2}, \quad \text{or } z = \frac{(m \log a \log b)^{\frac{1}{3}}}{(\log c)^{\frac{2}{3}}} - 1:$$

$$\text{similarly, } y = \frac{(m \log a \log c)^{\frac{1}{3}}}{(\log b)^{\frac{2}{3}}} - 1, \quad x = \frac{(m \log b \log c)^{\frac{1}{3}}}{(\log a)^{\frac{2}{3}}} - 1:$$

$$\text{also, } \frac{d^2u}{dx^2} = 2 \log c \,\frac{z+1}{(x+1)^2} = A,$$

$$\frac{d^2u}{dx\,dy} = \log c \,\frac{z+1}{(x+1)(y+1)} = B,$$

$$\frac{d^2u}{dy^2} = 2 \log c \,\frac{z+1}{(y+1)^2} = C:$$

wherefore A, C being positive, and AC greater than B^2, we are assured that the corresponding value of u will be a *minimum*.

285. If all the quantities A, B, C vanish, it will be necessary that the third terms of the developements of $u' - u$ and $u_{,} - u$ also vanish in the case of a maximum or a minimum as in functions of one variable, and its circumstances must be determined by means of the fourth terms of the said developements: also whenever any of the differential coefficients

assume the form $\frac{0}{0}$, or become infinite, they must be treated according to the principles applied in similar cases where only one variable is employed.

286. *To investigate the conditions necessary that a function of three variables may be a maximum or a minimum, and to deduce a criterion for deciding which it is.*

Retaining the notation of (269), and reasoning precisely as in (283), we shall find that the necessary conditions are comprised in the equations

$$\frac{du}{dx}=0,\quad \frac{du}{dy}=0,\quad \frac{du}{dz}=0:$$

and the third term of the developement of u' or $u_{,}$ is

$$\frac{d^2u}{dx^2}\frac{h^2}{1.2}+\frac{d^2u}{dxdy}hk+\frac{d^2u}{dy^2}\frac{k^2}{1.2}+\frac{d^2u}{dxdz}hl$$
$$+\frac{d^2u}{dydz}kl+\frac{d^2u}{dz^2}\frac{l^2}{1.2},$$

which may obviously be written

$$\frac{1}{1.2}\{Ah^2+2Bhk+Ck^2+2Dhl+2Ekl+Fl^2\},$$

if A, B, C, &c. represent the partial differential coefficients in order: and this again may assume the form

$$\frac{A}{1.2}(h+\frac{B}{A}k+\frac{E}{A}l)^2$$
$$+\frac{1}{1.2}\left\{\frac{AC-B^2}{A}k^2+\frac{2(D-E)A}{A}hl+\frac{2(A-B)E}{A}kl\right.$$
$$\left.+\frac{AF-E^2}{A}l^2\right\},$$

which will not change its algebraical sign whatever very small values be assigned to h, k, l, provided $(AC-B^2)(AF-E^2)$ be greater than $(AD-BE)^2$: and the function will therefore be a maximum or minimum only when $AC-B^2$ and $AF-E^2$

have the same algebraical sign, and according as A is negative or positive.

Ex. Find the maximum or minimum value of the function

$$u=(b^3-x^3)(x^2z-z^3)(xy-y^2).$$

Here $$\frac{du}{dx}=-3x^2(x^2z-z^3)(xy-y^2)+(b^3-x^3)(xy-y^2)\,2xz$$
$$+(b^3-x^3)(x^2z-z^3)\,y=0,$$
$$\frac{du}{dy}=(b^3-x^3)(x^2z-z^3)(x-2y)=0,$$
$$\frac{du}{dz}=(b^3-x^3)(x^2-3z^2)(xy-y^2)=0:$$

and from these we obtain the simultaneous values

$$x=\tfrac{1}{2}b\sqrt[3]{5},\ y=\tfrac{1}{4}b\sqrt[3]{5},\ z=\frac{1}{2\sqrt{3}}b\sqrt[3]{5},$$

which by the ordinary criterion may be proved to belong to a maximum.

287. If there be given one or more equations of condition to the proposed function, a process similar to that explained in (284) will furnish us with a number of equations sufficient for the solution: but it is readily observed that in functions of three or more variables the Determination and Verification of maxima and minima values will in general be attended with some degree of labour: and this may frequently be evaded by means of the following considerations.

Let $u=f(x, y, z, \&c.)$ where x, y, z, &c. are all entirely independent of each other: then supposing all the variables except one to have attained those determinate but unknown magnitudes which answer the condition, and differentiating on this hypothesis, we shall be enabled to express this one in terms of the rest: and proceeding similarly with regard to the others, we shall in general be furnished with the equations necessary for the determination of them all. This will appear best by examples.

Ex. 1. Find the maximum or minimum value of

$$u = \sin x \sin y \sin (a - x - y).$$

Supposing y to have attained its required value, and therefore to have become invariable, we have

$$\frac{du}{dx} = \cos x \sin y \sin (a - x - y) - \sin x \sin y \cos (a - x - y) = 0,$$

which gives immediately $\tan (a - x - y) = \tan x$:

$$\therefore\ a - x - y = x \text{ and } y = a - 2x:$$

whence by substitution we have

$$u = \sin^2 x \sin (a - 2x):$$

$$\therefore\ \frac{du}{dx} = 2 \sin x \cos x \sin (a - 2x) - 2 \sin^2 x \cos (a - 2x) = 0;$$

$$\therefore\ \tan (a - 2x) = \tan x, \text{ and } a - 2x = x \text{ or } x = \tfrac{1}{3} a:$$

consequently we have $y = a - 2x = \frac{1}{3} a$, and $u = (\sin \frac{1}{3} a)^3$ a *maximum*, since $\frac{d^2u}{dx^2}$ is then negative.

Ex. 2. Let $u = xyzt$ &c. be subject to the condition expressed by the equation

$$x + y + z + t + \&c. = a:$$

$$\text{then } du = \frac{du}{dx} dx + \frac{du}{dy} dy + \frac{du}{dz} dz + \frac{du}{dt} dt + \&c.:$$

but if we consider z, t, &c. to become invariable, we shall have $\frac{du}{dz} = 0$, $\frac{du}{dt} = 0$, &c.: and there then remains

$$du = \frac{du}{dx} dx + \frac{du}{dy} dy$$

$$= yzt \text{ \&c. } dx + xzt \text{ \&c. } dy = 0:$$

but by the equation of condition above given

$$dx + dy = 0 \text{ or } dy = -dx;$$

$$\therefore\ yzt \text{ \&c. } dx - xzt \text{ \&c. } dx = 0,$$

which shews that $y=x$: similarly we shall find $z=t=\&c.=x$: and if there be m such quantities we shall obviously have

$$x+y+z+t+\&c.=mx=a:$$

or $x=y=z=t=\&c.=\frac{a}{m}$, and the function $u=\left(\frac{a}{m}\right)^m$ will be a *maximum*, as is easily proved.

288. If in the equation $u=f(x, y, z, \&c.)$ the variables cease to be independent of one another, we shall have

$$\frac{d(u)}{dx}=\frac{du}{dx}+\frac{du}{dy}\frac{dy}{dx}+\frac{du}{dz}\frac{dz}{dx}+\&c.=0,$$

in case of a maximum or a minimum: and the same results might obviously be obtained from any of the equations

$$\frac{d(u)}{dy}=0, \quad \frac{d(u)}{dz}=0, \quad \&c.$$

289. If any of the partial differential coefficients employed in the preceding pages become of the indeterminate form $\frac{0}{0}$, it has been said that they must be treated according to the principles adopted in functions of one variable: that is, if the form $\frac{0}{0}$ occur when $x=a$ and $y=b$, we must substitute $a+h$ and $b+k$ in the places of x and y respectively, then develope both the numerator and denominator by the known methods, and finally assume $h=0$ and $k=0$: but this will not always give the true value of the expression unless we have some ratio assigned between the quantities h and k.

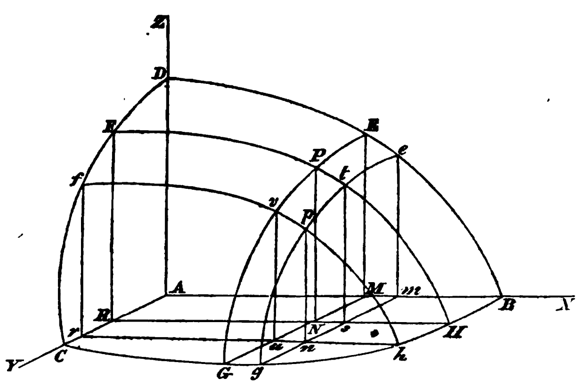
CHAP. XIV.

On the Application of the Differential Calculus to Curve Surfaces, and Curves of Double Curvature.

I. CURVE SURFACES.

290. Of Curve Surfaces referred to three rectangular axes, the general equation is usually expressed in one of the forms $z = f(x, y)$ or $f(x, y, z) = 0$ where x, y and z are the co-ordinates of any point: and the partial differential coefficients of z regarded as a function of x and y may be obtained according to the principles already laid down.

Hence if P be any point in the surface of the solid here represented, where the co-ordinate axes are AX, AY, AZ;



$AM = x$, $MN = y$, $NP = z$: and through it there be drawn two planes $FPHR$, $EPGM$ parallel to those of xz and zy respectively, the surface corresponding will be $DEPF$ whilst the portion of the volume of the solid defined by the same co-ordinates will be $APND$:

again, if the planes $fphr$, $epgm$ be drawn respectively parallel to these and $Mm=h$, $Rr=k$, $np=z+l$, we have from (259),

$$z+l=f(x+h,\ y+k)$$

$$=z+\frac{dz}{dx}h+\frac{d^2z}{dx^2}\frac{h^2}{1.2}+\&c.$$

$$+\frac{dz}{dy}k+\frac{d^2z}{dx\,dy}kh+\&c.$$

$$+\frac{d^2z}{dy^2}\frac{k^2}{1.2}+\&c.:$$

whereas had we supposed x to become $x+h$ and y to become $y+k$ separately, we should have found

$$st=z+\frac{dz}{dx}h+\frac{d^2z}{dx^2}\frac{h^2}{1.2}+\&c.$$

$$uv=z+\frac{dz}{dy}k+\frac{d^2z}{dy^2}\frac{k^2}{1.2}+\&c.:$$

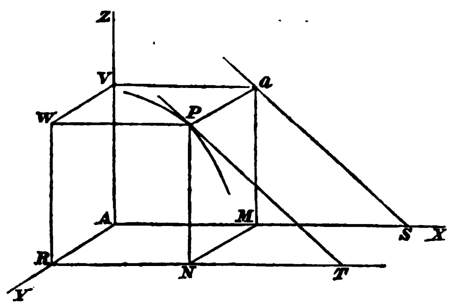
and these combined as in the article alluded to, will manifestly give the value of $z+l$ above exhibited, without any reference to the order in which the changes may have been made, whether AM be first changed in Am and then AR into Ar; or *vice versâ*.

If we put p and q for the values of the partial differential coefficients $\frac{dz}{dx}$ and $\frac{dz}{dy}$, we shall obviously have two differential equations $dz-p\,dx=0$ and $dz-q\,dy=0$ belonging to the sections of the surface respectively parallel to the planes of xz and yz.

Considerations and consequences arising from formulæ such as these will enable us to determine all the circumstances connected with surfaces defined by the said equation.

291. *To find the equation to a Plane touching a given surface at any assigned point.*

Let the equation to the plane touching the surface proposed at the point P be $z' = Ax' + By' + C$, where x', y', z' are the co-ordinates of the plane: then it is required only to express the values of A, B and C in terms of x, y, z, the co-ordinates of the proposed point of the surface:



now, since the plane passes through the point P, we must have

$$z = Ax + By + C;$$

$$\text{whence } z' - z = A(x' - x) + B(y' - y):$$

and it therefore remains only to find A and B.

Through P suppose a plane PNT to be drawn parallel to the plane of xz, and let it intersect the tangent plane at P in the line PT: then since at this point $y' = y$ the equation to the plane becomes

$$z' - z = A(x' - x),$$

in which we know that A is the trigonometrical tangent of the angle at which PT is inclined to NT, and therefore $A = \frac{dz}{dx}$:

similarly it may be shewn that $B = \frac{dz}{dy}$:

therefore the equation to the tangent plane now becomes

$$z' - z = \frac{dz}{dx}(x' - x) + \frac{dz}{dy}(y' - y),$$

which, if $\frac{dz}{dx} = p$ and $\frac{dz}{dy} = q$, is usually written in the form

$$z' - z = p(x' - x) + q(y' - y).$$

Since the point P is supposed to be assigned, we may consider x, y, z, p and q to be given magnitudes, and the equation may assume the ordinary form

$$z' = px' + qy' - (px + qy - z) = px' + qy' + (z - px - qy).$$

292. Cor. 1. Hence we may find the inclination of the tangent plane to each of the co-ordinate planes.

For, if α, β, γ denote the angles which the tangent plane makes with the co-ordinate planes of yz, xz, xy, and it pass through a point whose co-ordinates are x, y, z, its equation may assume the form

$$x\cos\alpha + y\cos\beta + z\cos\gamma = a:$$

$$\therefore\ p = \frac{dz}{dx} = -\frac{\cos\alpha}{\cos\gamma},\ q = \frac{dz}{dy} = -\frac{\cos\beta}{\cos\gamma}:$$

$$\text{whence } 1 + p^2 + q^2 = 1 + \frac{\cos^2\alpha}{\cos^2\gamma} + \frac{\cos^2\beta}{\cos^2\gamma} = \frac{1}{\cos^2\gamma}:$$

$$\therefore\ \cos\gamma = \frac{1}{\sqrt{1+p^2+q^2}} = \frac{1}{\sqrt{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2}}.$$

Hence also, $\cos\alpha = -\dfrac{p}{\sqrt{1+p^2+q^2}}$ and

$$\cos\beta = -\frac{q}{\sqrt{1+p^2+q^2}}.$$

293. Cor. 2. If it be required to draw to a curve surface a tangent plane which shall pass through a given point a, b, c, we shall have

$$c - z = p(a - x) + q(b - y),$$

which combined with the equation of the proposed surface, will produce an indeterminate equation of two variables affording the means of determining the points required of the surface through which the plane may pass.

Ex. 1. Let it be required to find the equation to the tangent plane of an ellipsoid whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Differentiating with respect to x and y in succession, we have

$$p=-\frac{c^2x}{a^2z} \text{ and } q=-\frac{c^2y}{b^2z}:$$

and substituting these values of p and q in the general equation to the tangent plane, we have

$$z'-z=-\frac{c^2x}{a^2z}(x'-x)-\frac{c^2y}{b^2z}(y'-y)$$

$$=-\frac{c^2xx'}{a^2z}+\frac{c^2x^2}{a^2z}-\frac{c^2yy'}{b^2z}+\frac{c^2y^2}{b^2z}:$$

whence multiplying both sides by $\frac{z}{c^2}$ and transposing, we obtain

$$\frac{zz'}{c^2}+\frac{xx'}{a^2}+\frac{yy'}{b^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}: \text{ or } \frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=1,$$

is the equation to the tangent plane.

To construct this plane, let $z'=0$:

$$\therefore \frac{xx'}{a^2}+\frac{yy'}{b^2}=1, \text{ or } y'=-\frac{b^2x}{a^2y}x'+\frac{b^2}{y},$$

is the equation to the intersection of the tangent plane with the plane of xy, which therefore makes with the axis of x an angle whose trigonometrical tangent is $-\frac{b^2x}{a^2y}$: and if y' be made $=0$, we have $x'=\frac{a^2y}{b^2x}\times\frac{b^2}{y}=\frac{a^2}{x}$, which is the rectangular subtangent measured along the axis of x:

similarly, the subtangents on the axes of y and z will be $\frac{b^2}{y}$ and $\frac{c^2}{z}$ respectively: and by means of the three points thus determined, or any two of them and the assigned point of the surface, the tangent plane may be constructed.

If it be required to draw a tangent plane to the ellipsoid passing through a given point whose co-ordinates are α, β, γ, we must obviously have

$$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2} = 1,$$

which, combined with the given equation to the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

will enable us to determine any two of the quantities x, y, z in terms of the remaining one, and thus to find the required points in the surface, which will therefore be subject to the equation of condition

$$\frac{(\alpha - x)\, x}{a^2} + \frac{(\beta - y)\, y}{b^2} + \frac{(\gamma - z)\, z}{c^2} = 0.$$

Ex. 2. Let the proposed surface be defined by the equation

$$\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a}:$$

$$\text{then } p = -\sqrt{\frac{z}{x}} \text{ and } q = -\sqrt{\frac{z}{y}}:$$

whence the equation to the tangent plane here becomes

$$\frac{x'}{\sqrt{x}} + \frac{y'}{\sqrt{y}} + \frac{z'}{\sqrt{z}} = \sqrt{a}:$$

and to enable us to construct it, we have the subtangents on the axes of x, y, z equal to $\sqrt{ax}$, $\sqrt{ay}$, $\sqrt{az}$ respectively.

Ex. 3. If $yz = \phi\left(\frac{y}{x}\right)$ be the equation to a curve surface; then will the part which the tangent plane at any point cuts off from the axis of z be equal to twice the value of z at that point.

Here $z = \frac{1}{y}\phi\left(\frac{y}{x}\right)$; whence by differentiation we have

$$p = -\frac{1}{x^2}\phi'\left(\frac{y}{x}\right) \text{ and } q = -\frac{1}{y^2}\phi\left(\frac{y}{x}\right) + \frac{1}{xy}\phi'\left(\frac{y}{x}\right);$$

and the equation to the tangent plane is

$$z' - z = -\frac{1}{x^2}\phi'\left(\frac{y}{x}\right)(x' - x) - \left\{\frac{1}{y^2}\phi\left(\frac{y}{x}\right) - \frac{1}{xy}\phi'\left(\frac{y}{x}\right)\right\}(y' - y):$$

wherefore, to obtain its intersection with the axis of z, we must make $x' = 0$ and $y' = 0$: whence will be found

$$z' - z = \frac{1}{x}\phi'\left(\frac{y}{x}\right) + \frac{1}{y}\phi\left(\frac{y}{x}\right) - \frac{1}{x}\phi'\left(\frac{y}{x}\right)$$

$$= \frac{1}{y}\phi\left(\frac{y}{x}\right) = z, \text{ or } z' = 2z.$$

294. *Of all the straight lines that can be drawn from the Point of Contact in the tangent plane, to find that which is inclined to the plane of* xy *at the greatest angle.*

Here it is manifest that the required line must be perpendicular to the intersection of the tangent plane with the plane of xy, and therefore its projection on that plane will also be perpendicular to it: but the equation to the tangent plane is

$$z' - z = p(x' - x) + q(y' - y),$$

from which, by making $z' = 0$, we obtain

$$y' - y = -\frac{p}{q}(x' - x) - \frac{z}{q}$$

for the equation of its intersection with the plane of xy:

let the equation of the straight line on the same plane perpendicular to this, be

$$y' = \frac{q}{p}x' + \mu:$$

then, since in this line y' becomes y when x' becomes x, we have

$$y = \frac{q}{p}x + \mu,$$

$$\text{whence } y' - y = \frac{q}{p}(x' - x):$$

and this equation, combined with the equation to the tangent plane, which may be put in the form

$$x' - x = -\frac{q}{p}(y' - y) + \frac{1}{p}(z' - z),$$

$$\text{gives } x' - x = -\frac{q^2}{p^2}(x' - x) + \frac{1}{p}(z' - z):$$

$$\text{whence } x' - x = \frac{p}{p^2 + q^2}(z' - z),$$

$$\text{and } \therefore y' - y = \frac{q}{p^2 + q^2}(z' - z),$$

are the equations to the required line of greatest inclination.

295. Cor. Hence the length of this line between the proposed point of the surface and the plane of xy will be the value of $\sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}$, when z' is made $=0$; which, with the equations just found, gives

$$L = z\sqrt{\frac{1 + p^2 + q^2}{p^2 + q^2}}:$$

and thence the angle of greatest inclination is obviously

$$= \sin^{-1}\frac{z}{L} = \sin^{-1}\sqrt{\frac{p^2 + q^2}{1 + p^2 + q^2}};$$

the co-ordinates of the angular point being

$$x' = x - \frac{pz}{p^2 + q^2} \text{ and } y' = y - \frac{qz}{p^2 + q^2},$$

as appears from making $z' = 0$, in the equations above found.

296. *To find the equations to the Normal of a curve surface at any proposed point.*

Let x, y, z be the co-ordinates of the point in the surface, $x_{,}, y_{,}, z_{,}$ those of any point in the normal: and suppose the equations of the normal to be

$$x_{,} = az_{,} + \mu \text{ and } y_{,} = \beta z_{,} + \nu,$$

in which it is required to find the values of a, μ, β, ν:

now, since the normal passes the point whose co-ordinates are x, y, z, we shall have

$$x = az + \mu \text{ and } y = \beta z + \nu;$$

whence we have likewise

$$x_{,} - x = a(z_{,} - z) \text{ and } y_{,} - y = \beta(z_{,} - z):$$

again, since the normal is perpendicular to the tangent plane, the projections of the normal must be perpendicular to the corresponding traces of the said plane: or the line belonging to

$$x_{,} - x = a(z_{,} - z)$$

must be perpendicular to the line belonging to

$$z' - z = p(x' - x) \text{ or } x' - x = \frac{1}{p}(z' - z):$$

whence we have $a = -p$: and similarly $\beta = -q$:

therefore the equations to the normal now become

$$x_{,} - x = -p(z_{,} - z) \text{ and } y_{,} - y = -q(z_{,} - z);$$

$$\text{or } x_{,} - x + p(z_{,} - z) = 0 \text{ and } y_{,} - y + q(z_{,} - z) = 0:$$

and by means of these the normal may be constructed.

297. Cor. 1. Let $x \cos \alpha + y \cos \beta + z \cos \gamma = a$, be the equation to the tangent plane; then are α, β, γ the angles which a normal or line perpendicular to it makes with the axes of x, y, z: whence, as in (292), we readily obtain

$$\cos \alpha = -\frac{p}{\sqrt{1+p^2+q^2}}, \quad \cos \beta = -\frac{q}{\sqrt{1+p^2+q^2}},$$

$$\text{and } \cos \gamma = \frac{1}{\sqrt{1+p^2+q^2}}.$$

298. Cor. 2. Hence we may find also the length of the normal, or the part of it intercepted between the surface and any of the co-ordinate planes.

The distance between two points x, y, z and $x_{\prime}$, $y_{\prime}$, $z_{\prime}$ being generally equal to

$$\sqrt{(x_{\prime}-x)^2+(y_{\prime}-y)^2+(z_{\prime}-z)^2},$$

if we suppose $x_{\prime}$, $y_{\prime}$ to become the co-ordinates of the point where the normal meets the plane of xy, we must manifestly make $z_{\prime}=0$ in the equations to the normal:

$$\text{that is, } x_{\prime} = x + pz \text{ and } y_{\prime} = y + qz,$$

will be the co-ordinates of the point of intersection: and the length of the normal to the plane of xy will therefore be

$$\sqrt{p^2z^2+q^2z^2+z^2} = z\sqrt{1+p^2+q^2}.$$

Similarly, the lengths of the normal to the planes of yz and xz may be found to be

$$\frac{x}{p}\sqrt{1+p^2+q^2} \text{ and } \frac{y}{q}\sqrt{1+p^2+q^2} \text{ respectively.}$$

299. Cor. 3. From this proposition we may draw to a curve surface a normal, which shall pass through a given point a, b, c.

For, since the equations to the normal at the point x, y, z of any surface, are

$$x_{\prime} - x + p(z_{\prime} - z) = 0 \text{ and } y_{\prime} - y + q(z_{\prime} - z) = 0;$$

we have $a - x + p(c - z) = 0$ and $b - y + q(c - z) = 0$:

whence by subtraction we obtain

$$x_{,} - a + p\,(z_{,} - c) = 0,$$
$$y_{,} - b + q\,(z_{,} - c) = 0,$$

and these, together with the equation to the surface proposed, will be sufficient for the determination of the point x, y, z: that is, since $z = f(x, y)$, $p = \phi(x, y)$ and $q = \psi(x, y)$, if $x_{,} = x$, $y_{,} = y$ and $z_{,} = z$, we shall have the three equations

$$z = f(x, y),$$
$$x - a + (z - c)\,\phi(x, y) = 0,$$
$$y - b + (z - c)\,\psi(x, y) = 0,$$

to find the three quantities x, y, z.

300. COR. 4. The Normal is the longest or shortest straight line that can be drawn from any point to the surface.

For, if u denote the square of the distance between the point whose co-ordinates are $x_{,}, y_{,}, z_{,}$, and the point x, y, z of the surface, then will

$$u = (x_{,} - x)^2 + (y_{,} - y)^2 + (z_{,} - z)^2,$$

in which z is a function of x and y: and we must have

$$\frac{du}{dx} = -(x_{,} - x) - (z_{,} - z)\frac{dz}{dx} = 0,$$

$$\frac{du}{dy} = -(y_{,} - y) - (z_{,} - z)\frac{dz}{dy} = 0,$$

by the nature of maxima and minima of two variables, as explained in (283): whence we derive immediately

$$x_{,} - x + \frac{dz}{dx}(z_{,} - z) = 0,$$

$$y_{,} - y + \frac{dz}{dy}(z_{,} - z) = 0,$$

which have been proved in (296) to be the equations of the normal, whose co-ordinates are $x_{,}, y_{,}, z_{,}$, to the surface at a point of which the co-ordinates are x, y, z.

Ex. Let the surface to which a normal is required to be drawn, be that of the elliptic paraboloid whose equation is

$$z = \frac{x^2}{a} + \frac{y^2}{b}:$$

$$\text{then } \frac{dz}{dx} = \frac{2x}{a} \text{ and } \frac{dz}{dy} = \frac{2y}{b}:$$

whence the equations to the normal become

$$x_{,} - x + \frac{2x}{a}(z_{,} - z) = 0,$$

$$y_{,} - y + \frac{2y}{b}(z_{,} - z) = 0:$$

and if $z_{,} = 0$, or the normal meet the plane of xy, we have

$$x_{,} = x + \frac{2xz}{a} \text{ and } y_{,} = y + \frac{2yz}{b}$$

for the co-ordinates of the point in which it meets that plane: also the length intercepted will be

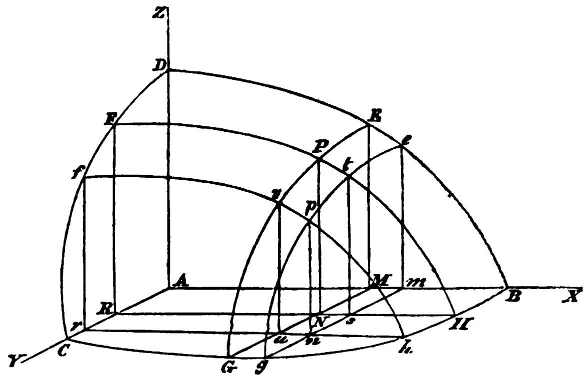
$$z\sqrt{1 + \frac{4x^2}{a^2} + \frac{4y^2}{b^2}} = \left(\frac{x^2}{a} + \frac{y^2}{b}\right)\sqrt{1 + \frac{4x^2}{a^2} + \frac{4y^2}{b^2}}.$$

If $a = b$, or the surface be that of the common paraboloid of revolution, then the length of the normal $= z\sqrt{1 + 4\frac{z}{a}}$, as it ought: for it is easily shown that the normal meets the axis of the solid, which is here the axis of z.

301. *To differentiate the Volume and Surface of a solid bounded by the co-ordinate planes and a curve surface, whose equation is* $z = f(x, y)$.

The construction being made as in (290), if V denote the volume $APND$, we shall obviously have $V = \phi(x, y)$: whence, if $Mm = h$, $Rr = k$ and V' be the Volume corres-

ponding to $x + h$ and $y + k$, it follows from *Taylor's* Theorem, that



$$V' - V = \frac{dV}{dx}h + \frac{d^2V}{dx^2}\frac{h^2}{1.2} + \&c.$$

$$+ \frac{dV}{dy}k + \frac{d^2V}{dx\,dy}hk + \&c.$$

$$+ \frac{d^2V}{dy^2}\frac{k^2}{1.2} + \&c.$$

$$+ \&c.:$$

also, if $'V$ and $''V$ represent the volumes corresponding to $x + h$ and y, x and $y + k$ separately considered, we shall have

$$'V - V = \frac{dV}{dx}h + \frac{d^2V}{dx^2}\frac{h^2}{1.2} + \&c.$$

$$''V - V = \frac{dV}{dy}k + \frac{d^2V}{dy^2}\frac{k^2}{1.2} + \&c.:$$

but, since $'V - V$ and $''V - V$ represent respectively the portions $PNme$ and $PNrf$ of the solid, it obviously follows that the remaining part of $V' - V$ must represent the remaining part of the increment of the solid corresponding to h and k the contemporaneous increments of x and y: whence we have

$$\frac{d^2 V}{dx\, dy} hk + \&c. = \text{the solid } PNnp:$$

$$\therefore \frac{d^2 V}{dx\, dy} + \&c. = \frac{\text{the solid } PNnp}{hk}:$$

and taking the limits of both sides of this equation, observing that the terms of the former side after the first vanish when h and k do, and that the solid $PNnp$ then becomes a prism whose base is hk, altitude z and volume therefore $= zhk$, we obtain $\dfrac{d^2 V}{dx\, dy} = z$ or $d^2 V = z\, dx\, dy$.

If the volume be supposed to be differentiated with respect to z as well as x and y, we shall obviously have

$$\frac{d^3 V}{dx\, dy\, dz} = 1 \text{ or } d^3 V = dx\, dy\, dz.$$

If S denote the surface of the solid, we shall find exactly in the same manner that

$$\frac{d^2 S}{dx\, dy} hk + \&c. = \text{the surface } Ptpv:$$

$$\therefore \frac{d^2 S}{dx\, dy} + \&c. = \frac{\text{the surface } Ptpv}{hk}:$$

and taking the limits as before observing that $Ptpv$ becomes coincident with the tangent plane, and is therefore equal to the base $Nn \times$ secant of the inclination of that plane to the plane of $xy = hk \sec \gamma = hk\sqrt{1 + p^2 + q^2}$ by (292), we have

$$\frac{d^2 S}{dx\, dy} = \sqrt{1 + p^2 + q^2} \text{ or } d^2 S = dx\, dy \sqrt{1 + p^2 + q^2}.$$

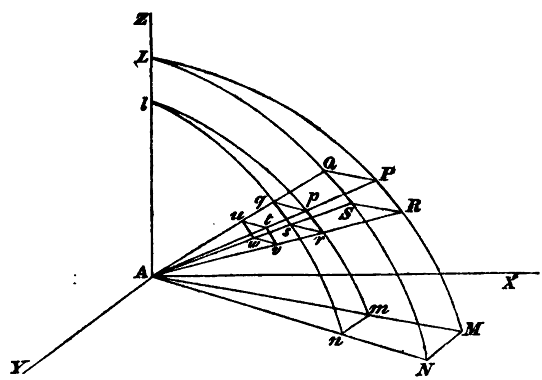
If the surface be differentiated with respect to all the co-ordinates, it readily follows as above that

$$\frac{d^3 S}{dx\,dy\,dz} = \frac{r+t}{\sqrt{1+p^2+q^2}} \text{ or } d^3 S = \frac{r+t}{\sqrt{1+p^2+q^2}}\,dx\,dy\,dz,$$

where r and t are the differential coefficients $\frac{d^2 z}{dx^2}$ and $\frac{d^2 z}{dy^2}$.

302. If polar co-ordinates be used, the differentials of the same magnitudes may easily be expressed in terms of them.

Let $APRSQ$ be a portion of the solid intercepted between the two planes $ALPRM$, $ALQSN$ through the axis of z perpendicular to the plane of xy, and the two planes APQ, ARS respectively perpendicular to these through the origin: the included angles being indefinitely small in each case:



and with centre A and radii Ap, At very nearly equal, suppose two spherical surfaces to be described cutting the solid: then if $Ap=\rho$, $XAM=\theta$ and $ZAP=\phi$, we shall have the elementary portion of the solid contained between these spherical surfaces = base × altitude $= pr \times pq \times pt$

$$= \rho\, d\phi \times \rho \sin\theta d\theta \times d\rho = \rho^2 \sin\theta\, d\theta\, d\rho\, d\phi:$$

$$\text{that is, } d^3 V = \rho^2 \sin\theta\, d\theta\, d\rho\, d\phi \text{ or } \frac{d^3 V}{d\theta\, d\rho\, d\phi} = \rho^2 \sin\theta.$$

Similarly the elementary portion of the surface of the solid between the planes above drawn

$$= PQRS = PQ \times PR = r^2 \sin\theta \, d\theta \, d\phi,$$

if $AP = r$: that is, $d^2 S = r^2 \sin\theta \, d\theta \, d\phi$

$$\therefore \frac{d^2 S}{d\theta \, d\phi} = r \sin^2\theta, \text{ and } \frac{d^3 S}{d\theta \, d\phi \, dr} = 2r \sin\theta.$$

It need scarcely be observed here that these conclusions have been deduced according to the principles of Infinitesimals explained in (92).

303. *To investigate the analytical circumstances of the Contacts and Osculations which may exist between two surfaces defined by given equations.*

If x, y, z be the co-ordinates of any point in a surface proposed, then when x and y become $x + h$ and $y + k$ respectively, the corresponding value of z will be expressed by

$$z + ph + qk + \frac{1}{1 \cdot 2}(rh^2 + 2shk + tk^2) + \&c.$$

where p, q, r, s, t, &c. are the partial differential coefficients taken in order: now if a surface of given species whose co-ordinates are x', y', z' pass through the point whose co-ordinates are x, y, z, it is required to ascertain the particular surface of this species which shall immediately about this point touch the proposed surface more closely than any other: but corresponding to the same increments h and k of the co-ordinates x' and y', we have the value of z' equal to

$$z' + Ph + Qk + \frac{1}{1 \cdot 2}(Rh^2 + 2Shk + Tk^2) + \&c.$$

P, Q, R, S, T, &c. denoting the partial differential coefficients in this case: whence making $x' = x$, $y' = y$ and $z' = z$, we shall obviously have the new value of z' in the same line with the new value of z, and the distance between the surfaces in the direction of this line will be expressed by

$$(P-p)\,h+(Q-q)\,k$$
$$+\frac{1}{1\,.\,2}\{(R-r)\,h^2+2\,(S-s)\,hk+(T-t)\,k^2\}+\&\text{c.}:$$

where x, y are supposed to be put for x', y' in the values of P, Q, R, S, T, &c.: whence if the constants which enter into the equation of the second surface be so determined that $P-p=0$ and $Q-q=0$, the two surfaces will have contact of the first order: if in addition to these $R-r=0$, $S-s=0$ and $T-t=0$, the surfaces will have contact of the second order; and so on: and it follows, that the greater the number of such equations is, the closer will be the contact immediately about the point proposed.

If we recollect, that besides these equations which are to determine the constants, there is one already established by changing x', y', z' into x, y, z, so that the touching surface may pass through the point x, y, z: it follows, that for an osculation of the first order three constants will be necessary and sufficient, six for one of the second, &c., $\frac{(n+1)(n+2)}{1\,.\,2}$ for one of the n^{th} order.

Ex. 1. Let the touching surface be a plane whose equation is

$$Ax'+By'+Cz'+1=0:$$

then, since this plane passes through the point x, y, z of the surface, we have

$$Ax+By+Cz+1=0:$$

$$\text{whence } z'-z=-\frac{A}{C}(x'-x)-\frac{B}{C}(y'-y):$$

$$\therefore\ \frac{dz'}{dx'}=-\frac{A}{C} \text{ and } \frac{dz'}{dy'}=-\frac{B}{C}:$$

$$\text{making therefore } p=-\frac{A}{C} \text{ and } q=-\frac{B}{C},$$

the equation to the plane becomes

$$z' - z = p(x' - x) + q(y' - y),$$

which is the equation to the tangent plane already found, and the tangent plane therefore osculates the surface at the point of contact.

Ex. 2. To determine the contacts which a sphere can have with any proposed surface.

Let α, β, γ be the co-ordinates of the centre of the sphere and δ the radius, then its equation is

$$(x' - \alpha)^2 + (y' - \beta)^2 + (z' - \gamma)^2 = \delta^2:$$

now, the equations $z' = z$, $P = p$ and $Q = q$, determine a contact of the first order; whence, if we differentiate this equation, we have

$$x' - \alpha + \frac{dz'}{dx'}(z' - \gamma) = 0,$$

$$y' - \beta + \frac{dz'}{dy'}(z' - \gamma) = 0:$$

and the three equations become now

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = \delta^2,$$

$$x - \alpha + p(z - \gamma) = 0,$$

$$y - \beta + q(z - \gamma) = 0:$$

which will be sufficient for the determination of any three of the quantities α, β, γ, δ.

The last two of these equations being those of a normal to the surface at a point whose co-ordinates are x, y, z, and passing through the point α, β, γ, it follows that the centre of the sphere is always situated in the normal drawn from the point of contact, and conversely.

Also, since we have three equations and four constants involved in them, it is manifest that one of them can be

assigned only in terms of the rest, so that the number of spheres having simple contact with the surface proposed is infinite. If the radius be given, the position of centre of the sphere is easily found.

If we suppose the contact to be of the second order, we must have likewise $R-r=0$, $S-s=0$ and $T-t=0$: that is, the six equations $z'=z$, $P-p=0$, $Q-q=0$, $R-r=0$, $S-s=0$, $T-t=0$, must be satisfied by the four quantities α, β, γ, δ, which cannot generally be the case: whence we conclude that a sphere is not generally capable of osculation, or of a complete contact of the second order, with any curve surface whatever.

304. Though it be impossible to define generally a sphere which shall have, with a proposed surface, osculation of either the first or second order, there is no difficulty in determining one which shall osculate with it in a given direction.

Let the proposed direction be defined by the equation $dy=mdx$; then it is obvious that $k=mh$, so that the term $\frac{1}{1.2}\{(R-r)h^2+2(S-s)hk+(T-t)k^2\}$ now becomes

$$\frac{1}{1.2}\{R-r+2(S-s)m+(T-t)m^2\}h^2,$$

which being put $=0$, will, together with the equations

$$z'-z=0,\ P-p=0 \text{ and } Q-q=0,$$

be sufficient for the determination of the four magnitudes α, β, γ, δ: the sphere may therefore have with the surface proposed a complete contact of the first order, and an incomplete contact of the second order taking place only in the direction of the section of the surface, whose tangent projected on the plane of xy, makes with the axis of x an angle of which the trigonometrical tangent is m.

To determine α, β, γ, δ in this case, we have

$$x'-\alpha+(z'-\gamma)P=0:$$

whence is obtained by differentiation with respect to x',

$$1 + P^2 + (z' - \gamma) R \quad \text{or} \quad R = -\frac{1 + P^2}{z' - \gamma}:$$

$$\text{similarly} \quad T = -\frac{1 + Q^2}{z' - \gamma} \quad \text{and} \quad S = -\frac{PQ}{z' - \gamma}:$$

changing now x', y', z', P, Q into x, y, z, p, q, we obtain

$$\frac{1 + p^2}{z - \gamma} + r + 2\left(\frac{pq}{z - \gamma} + s\right) m + \left(\frac{1 + q^2}{z - \gamma} + t\right) m^2 = 0:$$

from which is readily found

$$z - \gamma = -\frac{1 + p^2 + 2pqm + (1 + q^2) m^2}{r + 2sm + tm^2}.$$

Whence also $x - a = -p(z - \gamma)$, $y - \beta = -q(z - \gamma)$

and $\delta = (z - \gamma)\sqrt{1 + p^2 + q^2}$,

become known in terms of the co-ordinates of the proposed point, and thus the sphere is completely determined which osculates with the proposed surface solely in the direction of the curves, for which we have $dy = m dx$.

305. If it be required to find the values of m corresponding to a maximum or minimum value of δ, it is obvious that we must have $\frac{d\delta}{dm} = 0$:

and we have found above that

$$\delta = -\frac{1 + p^2 + 2pqm + (1 + q^2) m^2}{r + 2sm + tm^2}\sqrt{1 + p^2 + q^2},$$

which being differentiated with respect to m, the result made equal to zero and properly reduced, gives

$$\{(1 + q^2) s - pqt\} m^2 + \{r(1 + q^2) - t(1 + p^2)\} m$$
$$- \{(1 + p^2) s - pqr\} = 0,$$

from which the required values of m may be found: and thence we shall have

$$(rt-s)\,\delta^2+\{(1+p^2)\,t-2pqs+(1+q^2)\,r\}\sqrt{1+p^2+q^2}\,\delta$$
$$+(1+p^2+q^2)^2=0,$$

an equation by means of which the maximum and minimum value of the radius δ may be determined.

306. To simplify these expressions which are long and complicated, it is usual to transfer the origin of the co-ordinates to the point of contact under consideration, to make the axis of z coincident with the normal, and therefore the plane of xy coincident with the tangent plane: thus, we shall have $x=0$, $y=0$, $z=0$, $p=0$, $q=0$, and thence

$$\gamma=-\frac{1+m^2}{r+2sm+tm^2}:$$

$$\therefore\ \delta=\frac{1+m^2}{r+2sm+tm^2}:$$

and the equation for determining the values of m corresponding to a maximum and minimum of δ becomes

$$m^2+\left(\frac{r-t}{s}\right)m-1=0,$$

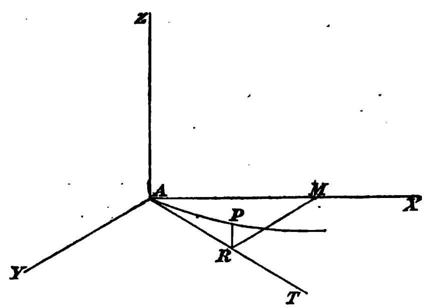
and if m_1 and m_2 be the values of m which satisfy this equation, we have $m_1m_2=-1$ or $m_1m_2+1=0$: so that if two planes pass through the normal and through the straight lines on the tangent plane defined by the equations $y=m_1x$ and $y=m_2x$, we get two circles, sections of the two spheres corresponding to the maximum and minimum of δ, and these sections are perpendicular to each other by virtue of the equation $m_1m_2+1=0$.

The osculations of the spheres take place along the curves which are the intersections of the proposed surface and the spheres by the cutting planes: and thus the great circles of the spheres are the osculating circles of the sections of the proposed

surface, otherwise there could some other circles pass between the curves on the surface and the said circles, and consequently there could be drawn spheres touching the proposed surface at this point more closely than the sphere above determined, which is impossible: whence the expression for δ is also the expression for the radius of curvature of the sections made on the proposed surface by the planes drawn through the normal, and the straight lines for which $\frac{dy}{dx} = m$. If we assume what has here been proved, we shall be enabled to establish by much simpler means the equations for m and δ already deduced: as in the following articles.

307. *To determine the Radius of Curvature of the section of a curve surface made by a plane passing through the normal at any point.*

Let AZ be the normal at the proposed point of the surface and the plane of xy a tangent plane there: also, let AR and



AP be the intersections of the tangent plane and the surface made by the proposed normal plane, the former having for its equation $y = mx$, and touching the latter at A: then if RP be drawn parallel to AZ, and δ be the radius of curvature at A of the plane curve AP, we shall have

$$\delta = \tfrac{1}{2} \text{ limit of } \frac{AP^2}{PR} = \tfrac{1}{2} \text{ limit of } \frac{AR^2}{PR}:$$

$$= \tfrac{1}{2} \text{ limit of } \frac{AM^2 + MR^2}{PR}:$$

but $PR = l = r\frac{h^2}{1.2} + shk + t\frac{k^2}{1.2} + \&c.$, since on this supposition $x = 0$, $y = 0$, $p = 0$ and $q = 0$:

$$\therefore \ \delta = \tfrac{1}{2} \text{ limit of } \frac{h^2 + k^2}{r\frac{h^2}{1.2} + shk + t\frac{k^2}{1.2} + \&c.}$$

$$= \text{limit of } \frac{1 + \left(\frac{k}{h}\right)^2}{r + 2s\frac{k}{h} + t\left(\frac{k}{h}\right)^2 + \&c.}$$

$$= \frac{1 + m^2}{r + 2sm + tm^2}, \text{ since } \frac{k}{h} = m:$$

whence if ϕ denote the angle which the normal section makes with the plane of xz, we shall have

$$\delta = \frac{1 + \tan^2\phi}{r + 2s\tan\phi + t\tan^2\phi}.$$

308. Cor. 1. If δ' be the radius of curvature of a section at right angles to the former, we shall have

$$\delta' = \frac{1 + \cot^2\phi}{r - 2s\cot\phi + t\cot^2\phi}:$$

whence is readily obtained

$$\frac{1}{\delta} + \frac{1}{\delta'} = r\cos^2\phi + t\sin^2\phi + r\sin^2\phi + t\cos^2\phi = r + t:$$

that is, the sum of the curvatures of any two normal sections at right angles to each other is a constant magnitude for the same given point.

309. Cor. 2. If we make $\phi = 0^0$ and $\phi = 90^0$ in succession, we shall have $\delta = \frac{1}{r}$ and $\delta' = \frac{1}{t}$ for the radii of curvature of the sections of the surface made by the planes of xz and yz respectively.

310. *To determine the normal Sections of greatest and least Curvature at any proposed point of a curve surface.*

We have already seen that at a given point of the surface, the curvature depends solely upon the quantity m since

$$\delta = \frac{1+m^2}{r+2sm+tm^2}:$$

$$\text{whence } \delta = \frac{1}{r\cos^2\phi + s\sin 2\phi + t\sin^2\phi}$$

must be a maximum or a minimum: let therefore

$$u = r\cos^2\phi + s\sin 2\phi + t\sin^2\phi,$$

and we have

$$\frac{du}{d\phi} = -2r\cos\phi\sin\phi + 2s\cos 2\phi + 2t\cos\phi\sin\phi = 0:$$

$$\therefore\ (t-r)\tan\phi + s(1-\tan^2\phi) = 0, \ldots\ldots\ldots\ldots(a)$$

whence we obtain

$$\tan\phi = \frac{t-r\pm\sqrt{(t-r)^2+4s^2}}{2s}:$$

therefore if ϕ_1 and ϕ_2 denote the angles made by the sections of greatest and least curvature with the plane of xz, we shall have

$$\tan\phi_1 = \frac{t-r+\sqrt{(t-r)^2+4s^2}}{2s},$$

$$\tan\phi_2 = \frac{t-r-\sqrt{(t-r)^2+4s^2}}{2s}:$$

which are therefore found.

$$\text{Also, since } \frac{1}{\delta} = r\cos^2\phi + s\sin 2\phi + t\sin^2\phi$$

$$= \sin^2\phi\,(r\cot^2\phi + 2s\cot\phi + t):$$

and from (a) we have

$$t = r - s\cot\phi + s\tan\phi;$$

$$\therefore\ r\cot^2\phi + 2s\cot\phi + t = r(1+\cot^2\phi) + s(\cot\phi + \tan\phi)$$

$$= r\operatorname{cosec}^2\phi + \frac{s}{\sin\phi\cos\phi}:$$

we obtain immediately

$$\frac{1}{\delta} = \sin^2\phi\left(r\operatorname{cosec}^2\phi + \frac{s}{\sin\phi\cos\phi}\right) = r + s\tan\phi$$

$$= r + \frac{t - r \pm \sqrt{(t-r)^2 + 4s^2}}{2} = \frac{t + r \pm \sqrt{(t-r)^2 + 4s^2}}{2}:$$

whence if δ_1 and δ_2 be the minimum and maximum radius of curvature, we have

$$\delta_1 = \frac{2}{t + r + \sqrt{(t-r)^2 + 4s^2}},$$

$$\delta_2 = \frac{2}{t + r - \sqrt{(t-r)^2 + 4s^2}},$$

which are therefore known.

311. Cor. From the last article we find $\tan\phi_1 \tan\phi_2 = -1$, from which we have

$$\tan\phi_2 = -\frac{1}{\tan\phi_1} = -\cot\phi_1 = \tan(90^\circ + \phi_1),$$

$$\text{and } \therefore\ \phi_2 = 90^\circ + \phi_1;$$

or the sections of greatest and least curvature are at right angles to each other.

312. *To express the Radius of Curvature of any normal section in terms of those of the normal sections of greatest and least curvature.*

Taking the point proposed as the origin of co-ordinates, and the normal as the axis of z, we may suppose the axes of x and y to be in the planes of greatest and least curvature, which are at right angles to each other: therefore the values of

m or of $\tan\phi$ which correspond to the sections of greatest and least curvature are 0 and ∞: but the equation

$$sm^2 + (r - t)\,m - s = 0,$$

gives $s = 0$ when $m = 0$; and the same equation expressed in the form

$$s + \left(\frac{r - t}{m}\right) - \frac{s}{m^2} = 0,$$

gives $s = 0$ when $m = \infty$, so that in both cases $s = 0$: hence we shall now have

$$\delta = \frac{1 + \tan^2\phi}{r + t\tan^2\phi} = \frac{\sec^2\phi}{r + t\tan^2\phi} = \frac{1}{r\cos^2\phi + t\sin^2\phi}:$$

but when $\phi = 0$, we have $\delta_1 = \dfrac{1}{r}$, and when $\phi = 90^0$, we have $\delta_2 = \dfrac{1}{t}$: therefore by substitution this expression becomes

$$\delta = \frac{1}{\dfrac{1}{\delta_1}\cos^2\phi + \dfrac{1}{\delta_2}\sin^2\phi} = \frac{\delta_1\delta_2}{\delta_1\sin^2\phi + \delta_2\cos^2\phi};$$

from which it follows that the radius of curvature of any normal section depends only upon the radii of greatest and least curvature, and upon the angle which the cutting plane makes with the sections to which these radii are the radii of curvature.

313. Cor. 1. Hence the curvatures of normal sections equally inclined to the plane of either greatest or least curvature are equal to one another.

314. Cor. 2. If we suppose $\delta_2 = \delta_1$, we shall have

$$\delta = \frac{\delta_1^2}{\delta_1(\sin^2\phi + \cos^2\phi)} = \delta_1,$$

which is independent of ϕ, and therefore every normal section passing through the proposed point will have the same curvature.

315. *To express the Radius of Curvature of the section of a surface made by any plane in terms of the greatest and least radii of curvature belonging to the same point.*

Let δ be the radius of curvature of the section made by a normal plane passing through the intersection of the proposed plane with the tangent plane, and suppose this plane to be inclined to the planes of greatest and least curvature at angles ϕ and $90° - \phi$, of which the radii of curvature are δ_1, δ_2: then we have

$$\delta = \frac{\delta_1 \delta_2}{\delta_1 \sin^2 \phi + \delta_2 \cos^2 \phi}:$$

now it is evident that the circle which is the intersection of the proposed plane with the sphere whose radius is δ is the circle of curvature of its intersection with the curve surface, and the radius of this circle $= \delta \sin \psi$, if ψ be the angle which the cutting plane makes with the tangent plane:

whence the radius of curvature of the proposed section

$$= \delta \sin \psi = \frac{\delta_1 \delta_2 \sin \psi}{\delta_1 \sin^2 \phi + \delta_2 \cos^2 \phi}.$$

316. Cor. If we suppose the angle $ZAx = \chi$, and the

radius of curvature of the oblique section to be δ', it is evident that $\psi = 90° - \chi$, and therefore $\delta' = \delta \cos \chi$.

Ex. 1. To find the radius of curvature of any normal section at the extremity of the axis of x of an ellipsoid.

If the origin be transferred to the proposed point, we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{(z-c)^2}{c^2}=1:$$

whence differentiating with respect to x and y, we have

$$\frac{x}{a^2}+\frac{z-c}{c^2}p=0,\quad \frac{1}{a^2}+\frac{p^2}{c^2}+\frac{z-c}{c^2}r=0:$$

$$\frac{y}{b^2}+\frac{z-c}{c^2}q=0,\quad \frac{1}{b^2}+\frac{q^2}{c^2}+\frac{z-c}{c^2}t=0:$$

$$\text{and also}\quad \frac{pq}{c^2}+\frac{z-c}{c^2}s=0,$$

but, since at the proposed point $x=0$, $y=0$, $z=0$, we have

$$p=0,\quad q=0,\quad \frac{1}{a^2}-\frac{r}{c}=0,\quad \frac{1}{b^2}-\frac{t}{c}=0\quad\text{and}\quad s=0;$$

whence we obtain immediately

$$\delta_1=\frac{1}{r}=\frac{a^2}{c}\quad\text{and}\quad \delta_2=\frac{1}{t}=\frac{b^2}{c},$$

which are the greatest and least radii of curvature at the point proposed: whence, for any other section through the axis of z, making an angle ϕ with the plane of xz, we shall have

$$\delta=\frac{\delta_1\delta_2}{\delta_1\sin^2\phi+\delta_2\cos^2\phi}=\frac{a^2b^2}{c\,(a^2\sin^2\phi+b^2\cos^2\phi)}.$$

If $a=b$, or the ellipsoid become an oblate spheroid, we obtain

$$\delta=\frac{a^2}{c\,(\sin^2\phi+\cos^2\phi)}=\frac{a^2}{c}:$$

which is the same for every normal section through the axis of z.

Ex. 2. To find the radius of curvature of any normal section of an oblate spheroid.

Let λ be the angle which a normal at the proposed point makes with the axis major, then it is easily shown that

$$\delta_1 = \frac{a(1-e^2)}{(1-e^2\sin^2\lambda)^{\frac{3}{2}}} \text{ and } \delta_2 = \frac{a}{\sqrt{1-e^2\sin^2\lambda}}:$$

whence by substitution we shall have

$$\delta = \frac{a}{\sqrt{1-e^2\sin^2\lambda}} \cdot \frac{1-e^2}{1-e^2+e^2\cos^2\alpha\cos^2\lambda},$$

if α be the angle made by the section with the generating ellipse.

317. *To find the equation of a normal Plane to a curve surface through a point whose co-ordinates are* x, y, z, *the section of the curve surface by this plane having* $dy = m\,dx$.

Let the equation to the required plane be

$$Ax' + By' + Cz' + 1 = 0:$$

$$\text{whence } Ax + By + Cz + 1 = 0,$$

$$\text{and } \therefore\ A(x'-x) + B(y'-y) + C(z'-z) = 0:$$

but, since the required plane passes through the normal whose equations are

$$x_, - x + p(z_, - z) = 0,$$

$$y_, - y + q(z_, - z) = 0,$$

we shall have by changing x' into $x_,$, y' into $y_,$, z' into $z_,$,

$$(z_, - z)\{C - Bq - Ap\} = 0:$$

which, as $z_,$ belongs to any point in the normal, gives

$$Ap + Bq - C = 0 \text{ or } \frac{A}{C}p + \frac{B}{C}q - 1 = 0:$$

$$\text{again, since } z' = -\frac{A}{C}x' - \frac{B}{C}y' - \frac{1}{C},$$

we have $\dfrac{d(z')}{dx'} = -\left(\dfrac{A}{C} + \dfrac{B}{C}\dfrac{dy'}{dx'}\right)$:

and proceeding in the plane from the point of contact along the section of the curve surface, we must have

$\dfrac{d(z')}{dx'} = \dfrac{d(z)}{dx}$: also, $\dfrac{dy'}{dx'} = \dfrac{dy}{dx} = m$, by hypothesis:

$$\therefore \frac{d(z)}{dx} = -\left\{\frac{A}{C} + \frac{B}{C}m\right\}:$$

whence, because $\dfrac{d(z)}{dx} = \dfrac{dz}{dx} + \dfrac{dz}{dy}\dfrac{dy}{dx} = p + qm$, we get

$$\frac{A}{C} + \frac{B}{C}m + p + qm = 0 \text{ or } \frac{A}{C}p + \frac{B}{C}mp + p^2 + pqm = 0:$$

but we have found $\dfrac{A}{C}p + \dfrac{B}{C}q - 1 = 0$:

therefore by subtraction and division we obtain

$$\frac{B}{C} = -\frac{p^2 + pqm + 1}{mp - q}:$$

also, since

$$\frac{A}{C}pm + \frac{B}{C}qm - m = 0 \text{ and } \frac{A}{C}q + \frac{B}{C}qm + pq + q^2m = 0,$$

we have by a similar process

$$\frac{A}{C} = \frac{m + pq + q^2m}{mp - q}:$$

whence the equation to the required plane now becomes

$$\frac{m + pq + q^2m}{mp - q}(x' - x) - \frac{p^2 + pqm + 1}{mp - q}(y' - y) + z' - z = 0:$$

and if for m there be substituted its proposed values, we can readily determine the greatest and least circles of curvature.

318. It is manifest that the value of m being a function of x, y, z, if we substitute this in the expressions for $x-\alpha$, $y-\beta$, $z-\gamma$, found in (304), and to these equations add the equation to the proposed surface, we shall have four equations, from which the quantities x, y, z being eliminated, there results an equation between α, β, γ, which will therefore be the equation to the surface which is the locus of the centres of all the spheres corresponding to $dy = m\,dx$: and if for m be substituted successively its values corresponding to the maximum and minimum radii of the spheres, we shall generally obtain a surface of two sheets, whereof one contains the centres of the spheres of the greatest radii and the other those of the spheres of the least.

319. *To determine a Paraboloid whose vertex shall osculate with a proposed surface at a given point.*

If the normal be considered as the axis of z and the axes of x and y be in the sections of greatest and least curvature, and $z = \dfrac{x^2}{2a} + \dfrac{y^2}{2b}$ be the equation to the paraboloid, then must a and b be the radii of greatest and least curvature of the surface at the given point: hence the equation to a normal section of the paraboloid inclined to the plane of xz at an angle ϕ, will become

$$z = (x^2+y^2)\left\{\frac{\sin^2\phi}{2b} + \frac{\cos^2\phi}{2a}\right\} = r^2\left(\frac{\sin^2\phi}{2b} + \frac{\cos^2\phi}{2a}\right),$$

which is that of a parabola whose *latus rectum* is

$$\frac{1}{\dfrac{\sin^2\phi}{2b} + \dfrac{\cos^2\phi}{2a}} = \frac{2ab}{a\sin^2\phi + b\cos^2\phi}:$$

the radius of curvature at whose vertex $= \dfrac{ab}{a\sin^2\phi + b\cos^2\phi}$; and this is also the expression for the radius of curvature of a normal section of the surface inclined to the plane of xz at the angle ϕ: that is, this paraboloid has with the proposed surface a complete contact of the second order.

320. Cor. Hence a very small portion of any curve surface whatever may be considered to possess the properties of a paraboloidal surface determined as above: also, since

$$\frac{x^2}{2az} + \frac{y^2}{2bz} = 1,$$

if we suppose z to be indefinitely diminished we find the ultimate section of every surface made by a plane parallel to the tangent plane to be a conic section, whose centre is situated in the normal, and whose axes lie in the planes of greatest and least curvature.

The curve thus obtained is sometimes called the *Indicatrix* of the surface proposed, and the points in which this curve becomes a circle are usually termed its *Umbilici*.

321. *To find the Directions on a curve surface in which the consecutive normals may intersect each other.*

Let the normal at any proposed point considered as the origin, be the axis of z; then the equations to the normal at a point whose co-ordinates are h, k, l, will be

$$x_{,} - h + p'(z_{,} - l) = 0, \quad y_{,} - k + q'(z_{,} - l) = 0:$$

where p', q' are the partial differential coefficients of the surface; therefore if the normal at this point intersect that at the origin, it is obvious that the two values of $z_{,} - l$ must be equal when $x_{,} = 0$ and $y_{,} = 0$: whence is readily found

$$\frac{p'}{h} = \frac{q'}{k}:$$

but by *Taylor's* theorem we have

$$p' = \frac{dp}{dx}h + \frac{dp}{dy}k + \&c.,$$

$$q' = \frac{dq}{dx}h + \frac{dq}{dy}k + \&c.,$$

since $p = 0$ and $q = 0$:

therefore by substitution is obtained

$$\frac{dp}{dx} + \frac{dp}{dy}m + \&c. = \frac{\frac{dq}{dx} + \frac{dq}{dy}m + \&c.}{m},$$

$$\text{or } r + sm + \&c. = \frac{s + tm + \&c.}{m}:$$

whence, if the points be consecutive, and therefore h and k be indefinitely small, this becomes

$$r + sm = \frac{s + tm}{m}, \quad \text{or} \quad m^2 + \left(\frac{r-t}{s}\right)m - 1 = 0,$$

an equation whose roots belong to the sections of maximum and minimum curvature, as appears from (309): and these are therefore the only two directions in which the intersections of consecutive normals can take place.

Hence we have likewise

$$z_{,} - l = \frac{h}{p'} = \frac{h}{\frac{dp}{dx}h + \frac{dp}{dy}k + \&c.} = \frac{1}{r + sm + \&c.},$$

which, when the points are consecutive, becomes

$$z_{,} = \frac{1}{r + sm}$$

$$= \frac{2}{2r + t - r \pm \sqrt{(t-r)^2 + 4s^2}} = \frac{2}{r + t \pm \sqrt{(t-r)^2 + 4s^2}};$$

that is, the distances from the surface at which these intersections take place are the radii of greatest and least curvature at the proposed point.

The loci of the points thus circumstanced are called the *Lines of Curvature* of the Surface, and every point of a surface is situated in the intersection of two curves of this description.

322. Cor. By means of this property the directions of the greatest and least curvatures of a surface may readily be determined.

Conical Surfaces.

323. Conical Surfaces may be considered to be produced by the successive intersections of a series of planes drawn according to some given law, and all passing through the same fixed point called the vertex.

Let a, β, γ be the co-ordinates of the fixed point; then the equation to one of these planes will be of the form

$$A(x-a) + B(y-\beta) + C(z-\gamma) = 0,$$

$$\text{or } \frac{A}{C}(x-a) + \frac{B}{C}(y-\beta) + z - \gamma = 0:$$

and its particular position will depend upon the quantities $\frac{A}{C}$ and $\frac{B}{C}$, which we may here represent by m and n:

but, since the planes are drawn according to some determinate law, m depends upon n so that we may suppose $m = f(n)$:

$$\therefore f(n)(x-a) + n(y-\beta) + z - \gamma = 0 \ldots\ldots (1):$$

and to pass to the consecutive plane we must obviously differentiate this equation with reference to n, considering x, y and z constant;

$$\therefore f'(n)(x-a) + y - \beta = 0:$$

$$\text{which gives } f'(n) = -\frac{y-\beta}{x-a}:$$

from which we may conclude that n is some function of $\frac{y-\beta}{x-a}$, as $n = \psi\left(\frac{y-\beta}{x-a}\right)$:

therefore equation (1) now becomes

$$(x-a)f\left\{\psi\left(\frac{y-\beta}{x-a}\right)\right\}+(y-\beta)\psi\left(\frac{y-\beta}{x-a}\right)+z-\gamma=0,$$

$$\text{or}\quad -f\left\{\psi\left(\frac{y-\beta}{x-a}\right)\right\}-\frac{y-\beta}{x-a}\psi\left(\frac{y-\beta}{x-a}\right)=\frac{z-\gamma}{x-a}:$$

and of this equation it is easy to see that the former member is a function of $\frac{y-\beta}{x-a}$, which may be represented by $\phi\left(\frac{y-\beta}{x-a}\right)$:

$$\text{whence we shall have } \frac{z-\gamma}{x-a}=\phi\left(\frac{y-\beta}{x-a}\right),$$

for the general equation to Conical Surfaces.

324. By eliminating the arbitrary function ϕ from this equation, we shall obtain the partial differential equation belonging to conical surfaces.

$$\text{For } \frac{dz}{dy}\,\frac{1}{x-a}=\phi'\left(\frac{y-\beta}{x-a}\right)\frac{1}{x-a},$$

$$\text{and } \frac{\frac{dz}{dx}(x-a)-(z-\gamma)}{(x-a)^2}=-\phi'\left(\frac{y-\beta}{x-a}\right)\frac{y-\beta}{(x-a)^2}:$$

and from the former we have

$$\phi'\left(\frac{y-\beta}{x-a}\right)=\frac{dz}{dy},$$

which substituted in the latter gives

$$\frac{dz}{dx}(x-a)-(z-\gamma)=-(y-\beta)\frac{dz}{dy}:$$

and this gives immediately

$$z-\gamma=\frac{dz}{dx}(x-a)+\frac{dz}{dy}(y-\beta),$$

for the partial differential equation to Conical Surfaces.

325. Cor. If the vertex be the origin of co-ordinates, then α, β, γ each $= 0$, and the equation becomes

$$\frac{x}{z} = \phi\left(\frac{y}{x}\right);$$

from which is readily deduced the partial differential equation

$$z = x\frac{dz}{dx} + y\frac{dz}{dy}.$$

326. The unknown function ϕ must be determined from the law according to which the series of planes is drawn.

Suppose the surface be required to pass through a given curve; then all the planes by the intersection of which the surface is formed will touch the given curve;

$$\text{therefore let } f(x, y, z) = 0 \ldots\ldots\ldots\ldots(1),$$

$$\text{and } F(x, y, z) = 0 \ldots\ldots\ldots\ldots(2),$$

be the equations to the two surfaces which contain the given curve, and assume

$$\frac{y-\beta}{x-\alpha} = M \ldots\ldots\ldots\ldots\ldots(3):$$

then, by the general equation to conical surfaces above found, we have

$$\frac{z-\gamma}{x-\alpha} = \phi(M) \ldots\ldots\ldots(4):$$

and from these four equations, which manifestly hold simultaneously, if x, y and z be eliminated, there will result an equation involving only M and $\phi(M)$, which consequently determines the form of the function ϕ:

whence, if for M and $\phi(M)$ their values be put in the resulting equation, the equation to the individual conical surface passing through the given curve will be found.

Ex. Let the generating curve, or the curve through which the conical surface passes, be a circle parallel to the plane of xy.

The equations to this circle are

$$(x-\alpha)^2+(y-\beta)^2=\gamma^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\text{and } z=c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2):$$

α and β being the co-ordinates of its centre, γ its radius and c its perpendicular distance from the plane of xy:

then if the vertex be the origin, the equation to the surface will be $\frac{z}{x}=\phi\left(\frac{y}{x}\right)$, as appears from (325):

$$\text{whence assuming } \frac{y}{x}=M \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$\text{we shall have } \frac{z}{x}=\phi(M) \ldots\ldots\ldots\ldots(4):$$

$$\therefore z=x\phi(M)=c,\ x=\frac{c}{\phi(M)} \text{ and } y=xM=\frac{cM}{\phi(M)};$$

which values of x and y being substituted in (1), give

$$\left(\frac{c}{\phi(M)}-\alpha\right)^2+\left(\frac{cM}{\phi(M)}-\beta\right)^2=\gamma^2:$$

and by the solution of this we may determine the form of the function ϕ;

whence substituting for M and $\phi(M)$ their values in terms of x, y and z we shall have

$$\left(\frac{cx}{z}-\alpha\right)^2+\left(\frac{cy}{z}-\beta\right)^2=\gamma^2,$$

which is the equation to the conical surface passing through the given circle.

If $\alpha=0$ and $\beta=0$, the equation just found becomes

$$x^2+y^2=\frac{\gamma^2}{c^2}z^2:$$

which is the equation to the right cone, the origin being the vertex and the axis of z coinciding with the axis of the cone.

If we suppose the circle through which the conical surface is to pass to be in the plane of xy and its centre at the origin, then the equations to this circle are, if a be its radius,

$$x^2 + y^2 = a^2 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\text{and } z = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2):$$

whence if the vertex be a point whose co-ordinates are α, β and γ, the equation to the surface is $\dfrac{z-\gamma}{x-\alpha} = \phi\left(\dfrac{y-\beta}{x-\alpha}\right)$:

$$\text{let therefore } \frac{y-\beta}{x-\alpha} = M \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$\text{then } \frac{z-\gamma}{x-\alpha} = \phi(M) \ldots\ldots\ldots\ldots\ldots\ldots (4):$$

and eliminating x, y and z from these four equations, we have

$$\left\{\alpha - \frac{\gamma}{\phi(M)}\right\}^2 + \left\{\beta - \frac{M\gamma}{\phi(M)}\right\}^2 = a^2,$$

for determining the form of the function ϕ: and restoring the values of M and $\phi(M)$, we obtain

$$\left\{\alpha - \gamma\frac{x-\alpha}{z-\gamma}\right\}^2 + \left\{\beta - \gamma\frac{y-\beta}{z-\gamma}\right\}^2 = a^2,$$

for the equation to the required surface.

If $\alpha = 0$ and $\beta = 0$, the surface will be that of a right cone, and the equation in that case becomes

$$x^2 + y^2 = \frac{a^2}{\gamma^2}(z-\gamma)^2.$$

327. *To find the equation to the Conical Surface which shall envelope another given surface.*

Here it will only be necessary to find the equations to the curve in which the two surfaces meet, and then the arbitrary function may be determined as before.

Now at every point in this curve it is obvious that the values of $\frac{dz}{dx}$ and $\frac{dz}{dy}$ must be the same for the given surface and the conical surface: substituting therefore in the partial differential equation to conical surfaces, the values of $\frac{dz}{dx}$ and $\frac{dz}{dy}$ deduced from the equation to the given surface, we shall thence obtain an equation $F(x, y, z)=0$: and this equation and the equation to the given surface will be the equations to the curve line in which the two surfaces meet, and therefore the equation to the individual conical surface which envelopes a given curve surface will be determined as before.

Ex. 1. Find the equation to the conical surface which shall envelope a paraboloid of revolution.

The equation to the paraboloid is $x^2+y^2=2az$, the vertex being the origin and $2a$ the *latus rectum:* hence the partial differential equation to the surface will be

$$z-\gamma=\frac{dz}{dx}(x-a)+\frac{dz}{dy}(y-\beta):$$

and for the points of contact of the two surfaces the partial differential coefficients $\frac{dz}{dx}$ and $\frac{dz}{dy}$ must be the same for each surface: but from the first equation,

$$x=a\frac{dz}{dx} \text{ or } \frac{dz}{dx}=\frac{x}{a},$$

$$y=a\frac{dz}{dy} \text{ or } \frac{dz}{dy}=\frac{y}{a}:$$

whence substituting these in the second equation, we have

$$z-\gamma=\frac{x}{a}(x-a)+\frac{y}{a}(y-\beta):$$

which with the equation $x^2+y^2=2az$ are the equations to the directrix: combining these two equations, we obtain

$$\alpha x+\beta y=a(z+\gamma)\ldots\ldots\ldots\ldots\ldots(1),$$

$$\text{and } x^2+y^2=2az\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

for the equations to the directrix, the former of which shows that the curve of contact of the two surfaces is a plane curve: now the equation to the conical surface being

$$\frac{z-\gamma}{x-\alpha}=\phi\left(\frac{y-\beta}{x-\alpha}\right),$$

$$\text{if we assume } \frac{y-\beta}{x-\alpha}=M\ldots\ldots\ldots\ldots(3),$$

$$\text{then will } \frac{z-\gamma}{x-\alpha}=\phi(M)\ldots\ldots\ldots\ldots(4):$$

and the elimination of x, y and z from these four equations will determine the form of the function ϕ: thus, let equations (1) and (2) be put under the forms

$$\alpha(x-\alpha)+\beta(y-\beta)+\alpha^2+\beta^2=a(z-\gamma)+2a\gamma,$$

$$\text{and } (x-\alpha)^2+(y-\beta)^2+2\alpha(x-\alpha)+2\beta(y-\beta)$$
$$+\alpha^2+\beta^2=2a(z-\gamma)+2a\gamma:$$

then subtracting twice the former from the latter, we have

$$(x-\alpha)^2+(y-\beta)^2=\alpha^2+\beta^2-2a\gamma:$$

substituting in this and the last but one, the values of $y-\beta$ and $z-\gamma$ from equations (3) and (4), we get

$$\alpha(x-\alpha)+\beta M(x-\alpha)+\alpha^2+\beta^2=a\phi(M)(x-\alpha)+2a\gamma,$$

$$\text{and } (x-\alpha)^2+(x-\alpha)^2M^2=\alpha^2+\beta^2-2a\gamma:$$

equating the values of $x-\alpha$ from these two equations, we find

$$\frac{\alpha^2+\beta^2-2a\gamma}{1+M^2}=\left(\frac{\alpha^2+\beta^2-2a\gamma}{a\phi(M)-\beta M-\alpha}\right)^2,$$

$$\text{or } \{a\phi(M)-\beta M-\alpha\}^2=(1+M^2)(\alpha^2+\beta^2-2a\gamma),$$

which determines the form of the function ϕ: then restoring the values of M and $\phi(M)$, we have

$$\left\{a\frac{z-\gamma}{x-a}-\beta\frac{y-\beta}{x-a}-a\right\}^2=\left\{1+\left(\frac{y-\beta}{x-a}\right)^2\right\}(a^2+\beta^2-2a\gamma),$$

$$\text{or } \{a(z-\gamma)-\beta(y-\beta)-a(x-a)\}^2$$

$$=\{(x-a)^2+(y-\beta)^2\}(a^2+\beta^2-2a\gamma),$$

which is the equation to the required enveloping surface.

When $a=0$, $\beta=0$ and γ is negative, the enveloping surface will evidently be a right cone whose axis coincides with the axis of z, the equation in that case being

$$a(z+\gamma)=\sqrt{x^2+y^2}\sqrt{2a\gamma} \text{ or } x^2+y^2=\frac{1}{2}\frac{a}{\gamma}(z+\gamma)^2.$$

Ex. 2. Find the conical surface which shall envelope a surface of the second order whose equation is

$$Ax^2+By^2+Cz^2+2Gx+2Hy+2Kz+L=0.$$

Here $\dfrac{dz}{dx}=-\dfrac{Ax+G}{Cz+K}$ and $\dfrac{dz}{dy}=-\dfrac{By+H}{Cz+K}$: which substituted in the general equation

$$\gamma-z=\frac{dz}{dx}(a-x)+\frac{dz}{dy}(\beta-y), \text{ give}$$

$$\gamma-z=-\frac{Ax+G}{Cz+K}(a-x)-\frac{By+H}{Cz+K}(\beta-y),$$

$$\text{or } Ax^2+By^2+Cz^2+Gx+Hy+Kz$$

$$-Aax-B\beta y-C\gamma z-Ga-H\beta-K\gamma=0:$$

this and the equation to the given surface, are the equations of the surfaces whose intersection is the directrix: whence also any combination of these two equations are equally the equations of the directrix: therefore by subtraction we have

$$(G+Aa)x+(H+B\beta)y+(K+C\gamma)z+Ga+H\beta+K\gamma+L=0,$$

which is the equation to a plane, and therefore the curve of contact of the two surfaces is a plane curve:

if we take the vertex for the origin, then $\alpha = 0$, $\beta = 0$, $\gamma = 0$, and the latter equation becomes

$$Gx + Hy + Kz + L = 0:$$

subtracting twice this equation from that to the given surface, we have for the equations to the directrix

$$Ax^2 + By^2 + Cz^2 - L = 0 \ldots\ldots\ldots\ldots\ldots (1),$$

$$Gx + Hy + Kz + L = 0 \ldots\ldots\ldots\ldots\ldots (2):$$

and the equation to the conical surface will be obtained by eliminating M from the two equations

$$\frac{y}{x} = M \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$\frac{z}{x} = \phi(M) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4):$$

whence it follows that x, y and z must also be eliminated:

now $x = \dfrac{z}{\phi(M)}$ and $y = xM = \dfrac{Mz}{\phi(M)}$, which substituted in (2) give

$$G\frac{z}{\phi(M)} + H\frac{Mz}{\phi(M)} + Kz + L = 0:$$

$$\text{whence } z = -\frac{L}{K + H\dfrac{M}{\phi(M)} + \dfrac{G}{\phi(M)}} = -\frac{L\phi(M)}{G + HM + K\phi(M)}:$$

$$\therefore x = \frac{z}{\phi(M)} = -\frac{L}{G + HM + K\phi(M)},$$

$$\text{and } y = \frac{Mz}{\phi(M)} = -\frac{LM}{G + HM + K\phi(M)}:$$

and substituting these values of x, y and z in (1), we have

$$\{G+HM+K\phi(M)\}^2=L\{A+BM^2+C\overline{\phi(M)}|^2\},$$

which determines the form of the function ϕ; whence restoring the values of M and $\phi(M)$, we obtain

$$(Gx+Hy+Kz)^2=L(Ax^2+By^2+Cz^2)$$

for the equation to the required enveloping surface.

Cylindrical Surfaces.

328. Cylindrical Surfaces may be conceived to be produced by the successive intersections of a series of planes, drawn according to some known law, and always perpendicular to a given plane.

Let the equation to the given plane be

$$ax'+by'+cz'=0,$$

and let the equation to any one of the series of planes be

$$Ax+By+Cz+1=0:$$

then these two planes being perpendicular to each other, we have the equation of condition

$$Aa+Bb+Cc=0:$$

whence, if we make $\frac{A}{C}=m$ and $\frac{B}{C}=n$, we shall have

$$n=-\frac{c+am}{b};$$

now, putting the equation $Ax+By+Cz+1=0$, under the form

$$\frac{A}{C}x+\frac{B}{C}y+z+\frac{1}{C}=0,$$

$$\text{or } mx+ny+z+\frac{1}{C}=0,$$

and substituting for n its value, we obtain

$$m(bx - ay) + bz - cy + \frac{b}{C} = 0:$$

and the position of this plane depends only on the two quantities m and $\frac{b}{C}$; but since the planes are drawn according to some known law, $\frac{b}{C}$ is a function of m; let, therefore, $\frac{b}{C} = \phi(m)$, which gives

$$m(bx - ay) + bz - cy + \phi(m) = 0:$$

and to pass to the consecutive plane, we must obviously differentiate this equation with respect to m, considering x, y and z as constant:

$$\therefore\ bx - ay + \phi'(m) = 0:$$

which shews that m is a function of $bx - ay$:

supposing, therefore, $m = \psi(bx - ay)$, and substituting this value in the last equation but one, we have

$$(bx - ay)\,\psi(bx - ay) + bz - cy + \phi\{\psi(bx - ay)\} = 0:$$

which equation proves that $bz - cy$ is a function of $bx - ay$;

and therefore $bz - cy = f(bx - ay)$

will be the general equation to Cylindrical Surfaces.

329. To obtain the partial differential equation of cylindrical surfaces.

Differentiating with respect to x only, we have

$$b\frac{dz}{dx} = f'(bx - ay)\,b:$$

and differentiating with respect to y only, we get

$$b\frac{dz}{dy} - c = f'(bx - ay)(-a):$$

whence, eliminating $f'(bx - ay)$ from these two equations, we obtain

$$a\frac{dz}{dx} + b\frac{dz}{dy} - c = 0$$

for the partial differential equation to Cylindrical Surfaces.

Ex. Let the directrix be a circle parallel to the plane of xy, α, β, γ being the co-ordinates of its centre and δ its radius: then the equations to this circle are

$$(x-\alpha)^2 + (y-\beta)^2 = \delta^2 \text{.........}(1),$$

$$\text{and } z = \gamma \text{.........}(2):$$

and the equation to the cylindrical surface here results from the elimination of M between the two equations

$$bx - ay = M \text{...............}(3),$$

$$\text{and } bz - cy = \phi(M) \text{..........}(4);$$

let x, y and z be eliminated from these four equations; then

$$y = \frac{1}{c}\{bz - \phi(M)\} = \frac{b\gamma - \phi(M)}{c}:$$

$$\therefore x = \frac{ay + M}{b} = \frac{a}{bc}\{b\gamma - \phi(M)\} + \frac{M}{b}:$$

substituting these values of x and y in equation (1), we have

$$\left\{\frac{ab\gamma - a\phi(M) + cM}{bc} - \alpha\right\}^2 + \left\{\frac{b\gamma - \phi(M)}{c} - \beta\right\}^2 = \delta^2,$$

which determines the form of the function ϕ: whence, restoring the values of M and $\phi(M)$, we have

$$\left\{\frac{ab\gamma - abz + acy + bcx - acy}{bc} - a\right\}^2 + \left\{\frac{b\gamma - bz + cy}{c} - \beta\right\}^2 = \delta^2,$$

$$\text{or } (a\gamma - ca - az + cx)^2 + (b\gamma - c\beta - bz + cy)^2 = c^2\delta^2,$$

which is the equation to the required surface.

If $a = 0$ and $\beta = 0$, the surface will then evidently become that of the right cylinder, and the equation reduces itself to

$$\{a(\gamma - z) + cx\}^2 + \{b(\gamma - z) + cy\}^2 = c^2\delta^2, \text{ or}$$

$$a^2(\gamma - z)^2 + b^2(\gamma - z)^2 + (2acx + 2bcy)(\gamma - z) + c^2x^2 + c^2y^2 = c^2\delta^2:$$

but also in this case $a = 0$ and $b = 0$, since the plane to which the series of generating planes are drawn perpendicular is parallel to the plane of xy; these values of a and b reduce the equation to

$$x^2 + y^2 = \delta^2,$$

which is the equation to a right cylinder, the origin being some point of its axis.

330. *To find the equation to the Cylindrical Surface which shall envelope a given ellipsoid.*

Let the equation to the ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1:$$

$$\text{then } \frac{x}{a^2} + \frac{z}{\gamma^2}\frac{dz}{dx} = 0, \quad \therefore \frac{dz}{dx} = -\frac{\gamma^2}{a^2}\frac{x}{z}:$$

$$\text{and } \frac{y}{\beta^2} + \frac{z}{\gamma^2}\frac{dz}{dy} = 0, \quad \therefore \frac{dz}{dy} = -\frac{\gamma^2}{\beta^2}\frac{y}{z}:$$

substituting these values in the partial differential equation to a cylindrical surface, viz. in the equation

$$a\frac{dz}{dx} + b\frac{dz}{dy} = c,$$

we have

$$-a\frac{\gamma^2}{\alpha^2}\frac{x}{z}-b\frac{\gamma^2}{\beta^2}\frac{y}{z}=c, \text{ or } \frac{cz}{\gamma^2}+\frac{by}{\beta^2}+\frac{ax}{\alpha^2}=0:$$

which is the equation to a plane passing through the origin, and therefore the curve of contact of the two surfaces is a plane curve, of which the last equation and that of the given surface are the equations: hence the four following equations hold simultaneously for the points of contact of the two surfaces:

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1 \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\frac{ax}{\alpha^2}+\frac{by}{\beta^2}+\frac{cz}{\gamma^2}=0 \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$bx-ay=M \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$bz-cy=\phi(M) \ldots\ldots\ldots\ldots\ldots\ldots(4):$$

from which x, y and z are to be eliminated; thus

$$x=\frac{ay+M}{b} \text{ and } z=\frac{cy+\phi(M)}{b};$$

substituting these values in (2) we have

$$\frac{a}{b\alpha^2}(ay+M)+\frac{by}{\beta^2}+\frac{c}{\gamma^2 b}\{cy+\phi(M)\}=0,$$

$$\text{or } \left(\frac{a^2}{b\alpha^2}+\frac{b}{\beta^2}+\frac{c^2}{b\gamma^2}\right)y+\frac{c\phi(M)}{b\gamma^2}+\frac{aM}{b\alpha^2}=0:$$

$$\text{whence } y=-\frac{\frac{c}{\gamma^2}\phi(M)+\frac{a}{\alpha^2}M}{\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}}:$$

$$x=-\frac{\frac{ac}{b\gamma^2}\phi(M)+\frac{a^2}{b\alpha^2}M}{\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}}+\frac{M}{b}=\frac{1}{b}\,\frac{\left(\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)M-\frac{ac}{\gamma^2}\phi(M)}{\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}}:$$

$$\text{and } z = \frac{c}{b}y + \frac{1}{b}\phi(M) = -\frac{\dfrac{c^2}{b\gamma^2}\phi(M) + \dfrac{ac}{b\alpha^2}M}{\dfrac{a^2}{\alpha^2} + \dfrac{b^2}{\beta^2} + \dfrac{c^2}{\gamma^2}} + \frac{\phi(M)}{b}$$

$$= \frac{1}{b}\,\frac{\left(\dfrac{a^2}{\alpha^2} + \dfrac{b^2}{\beta^2}\right)\phi(M) - \dfrac{ac}{\alpha^2}M}{\dfrac{a^2}{\alpha^2} + \dfrac{b^2}{\beta^2} + \dfrac{c^2}{\gamma^2}}:$$

and if these values of x, y and z be substituted in (1), we get

$$\left\{\left(\frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2}\right)\frac{M}{b\alpha} - \frac{ac}{\alpha\gamma^2}\frac{\phi(M)}{b}\right\}^2 + \left\{\frac{c}{\gamma^2}\frac{\phi(M)}{\beta} + \frac{a}{\alpha^2}\frac{M}{\beta}\right\}^2$$

$$+ \left\{\left(\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2}\right)\frac{\phi(M)}{\gamma b} - \frac{ac}{\alpha^2}\frac{M}{\gamma b}\right\}^2 = \left(\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2}\right)^2:$$

which determines the form of the function ϕ: restoring the values of M and $\phi(M)$, we then have

$$\left\{\left(\frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2}\right)\frac{bx-ay}{b\alpha} - \frac{ac}{\gamma^2}\frac{bz-cy}{b\alpha}\right\}^2 + \left\{\frac{c}{\gamma^2\beta}(bz-cy) + \frac{a}{\alpha^2\beta}(bx-ay)\right\}^2$$

$$+ \left\{\left(\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2}\right)\frac{bz-cy}{b\gamma} - \frac{ac}{\alpha^2}\frac{bx-ay}{b\gamma}\right\}^2 = \left\{\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2}\right\}^2,$$

which is the equation to the surface required.

If the surface be perpendicular to the plane of xy, we have $a = 0$ and $b = 0$, which reduces the last equation to

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1;$$

which is that of an elliptic right cylinder.

Surfaces of Revolution.

331. A Surface of Revolution may be conceived to be formed by the successive intersections of a series of spheres

having their centres in the same straight line called the axis, the radius of each sphere being some function of the co-ordinates of its centre.

Let the equations of the straight line be

$$x' = az' + \alpha \text{ and } y' = bz' + \beta;$$

then the equation to a sphere having its centre on this line will be

$$(x - x')^2 + (y - y')^2 + (z - z')^2 = \delta^2,$$

where x, y, z are the co-ordinates of any point of the sphere; or by the substitution of the values of x' and y',

$$(x - az' - \alpha)^2 + (y - bz' - \beta)^2 + (z - z')^2 = \delta^2:$$

but δ is some function of x', y' and z' and therefore some function of z' since x' and y' are functions of z': whence

$$(x - az' - \alpha)^2 + (y - bz' - \beta)^2 + (z - z')^2 = \phi(z'),$$

is the equation to any one of the spheres by which the surface is generated: and to find the intersection of any two consecutive spheres we must differentiate considering z' as variable, and x, y, z as constant, since these three belong to both spheres: hence we have the equation

$$(x - az' - \alpha)\, a + (y - bz' - \beta)\, b + (z - z') - \phi'(z') = 0:$$

between which and the one last found z' must obviously be eliminated in order to have the equation to the curve which is the locus of all such successive intersections: transposing in the last, we have

$$\begin{aligned}(x - \alpha)\, a + (y - \beta)\, b + z &= \phi'(z') + a^2 z' + b^2 z' + z' \\ &= \phi_1(z'):\end{aligned}$$

transposing in the former we obtain

$$\begin{aligned}&(x - \alpha)^2 + (y - \beta)^2 + z^2 \\ &= \phi(z') + 2az'(x - \alpha) + 2bz'(y - \beta) + 2zz' - (a^2 + b^2 + 1)\, z'^2 \\ &= \phi(z') + 2z'\phi_1(z') - (a^2 + b^2 + 1)\, z'^2 \\ &= \phi_2(z'):\end{aligned}$$

$$\therefore\ (x-a)^2+(y-\beta)^2+z^2=f\left\{(x-a)\,a+(y-\beta)\,b+z\right\},$$

is the general equation of Surfaces of Revolution.

332. To find the partial differential equation, we must differentiate and eliminate the arbitrary function: thus we get

$$2\,(x-a)+2z\frac{dz}{dx}=f'\left\{(x-a)\,a+(y-\beta)\,b+z\right\}\left(a+\frac{dz}{dx}\right),$$

also $2\,(y-\beta)+2z\dfrac{dz}{dy}=f'\left\{(x-a)\,a+(y-\beta)\,b+z\right\}\left(b+\dfrac{dz}{dy}\right);$

whence, by elimination, we obtain immediately

$$(x-a+pz)\,(b+q)=(y-\beta+qz)\,(a+p),$$

or $(x-a)\,(b+q)-(y-\beta)\,(a+p)+pbz+pqz-aqz-pqz=0,$

that is, $(x-a)\,(b+q)-(y-\beta)\,(a+p)+z\,(pb-aq)=0.$

If the axis of revolution coincide with the axis of z, we shall have

$$a=0,\ b=0,\ a=0,\ \beta=0;$$

and the equation then becomes

$$x^2+y^2+z^2=\phi\,(z)\ \text{ or }\ x^2+y^2=\phi\,(z)-z^2;$$

that is, $x^2+y^2=\psi\,(z)$, or $z=f\,(x^2+y^2)$:

and in this case the partial differential equation becomes

$$qx-py=0.$$

Annular Surfaces.

333. ANNULAR Surfaces may be supposed to be generated by the successive intersections of a series of spheres of equal radii having their centres situated in a given curve.

Let the curve be in the plane of xy, then the equation to a sphere at the points x', y' in the plane of xy is

$$(x-x')^2+(y-y')^2+z^2=a^2:$$

but the co-ordinates of the point x', y' being subject to satisfy the equation of the given curve, one is a function of the other, so that if m be any particular value of x', we must have $y'=\phi(m)$:

$$\therefore\ (x-m)^2+\{y-\phi(m)\}^2+z^2=a^2 \text{.............}(1),$$

is the equation to one particular sphere, and to pass to the consecutive one, we must differentiate with respect to m, considering x, y, z as constant; whence

$$x-m+\{y-\phi(m)\}\ \phi'(m)=0 \text{............}(2):$$

these two equations therefore express the relations existing between the co-ordinates x, y, z of the points of intersection of the two spheres, and the elimination of m would give the surface which is the locus of all such points; but m cannot here be eliminated since a value of $\phi(m)$ in terms of x, y and z cannot be obtained from (2), which shows, however, that m is some unknown function of x, y, z: differentiating (1) first with respect to x and z, and then with regard to y and z, considering m variable but subject to the condition expressed by (2), which is evidently the same thing as differentiating (1) considering m constant, we have

$$x-m+pz=0,$$

$$y-\phi(m)+qz=0:$$

and these two equations express all the conditions comprised in (1) and (2) as is evident from what has been said: then substituting the values of $x-m$, $y-\phi(m)$ found from these in (1), we have

$$z^2(1+p^2+q^2)=a^2,$$

which is the partial differential equation of Annular Surfaces.

334. Cor. Since this equation gives

$$z = \frac{a}{\sqrt{1+p^2+q^2}},$$

and $\dfrac{1}{\sqrt{1+p^2+q^2}}$ expresses the value of the cosine of the angle made by the normal with the axis of z, if n be the length of the part of the normal intercepted between the surface and the plane of xy, we have

$$z = \frac{n}{\sqrt{1+p^2+q^2}}:$$

whence it follows that $n = a$, or the normal at any point of an annular surface is equal to the radius of the sphere, as it ought.

Ex. Let the curve which is the locus of the centres of the spheres be a circle, whose equation is

$$x'^2 + y'^2 = c^2:$$

then, since m and $\phi(m)$ are corresponding values of x' and y', we have

$$m^2 + \{\phi(m)\}^2 = c^2 \ldots\ldots\ldots\ldots (1):$$

and this, with the two equations

$$(x-m)^2 + \{y - \phi(m)\}^2 + z^2 = a^2 \ldots\ldots (2),$$

$$x - m + \{y - \phi(m)\}\,\phi'(m) = 0 \ldots\ldots\ldots (3),$$

will manifestly lead to the equation of the required surface by the elimination of m:

differentiating (1) we have

$$m + \phi(m)\,\phi'(m) = 0;$$

$$\text{whence } \phi'(m) = -\frac{m}{\phi(m)}:$$

and substituting this in (3), we get

$$x - m + \{y - \phi(m)\} \frac{-m}{\phi(m)} = 0 \quad \text{or} \quad x - m - \frac{my}{\phi(m)} + m = 0;$$

from which is readily obtained $\phi(m) = \dfrac{my}{x}$,

which substituted in (1) gives

$$m^2 + \frac{m^2 y^2}{x^2} = c^2:$$

whence $m = \dfrac{cx}{\sqrt{x^2+y^2}}$ and $\therefore\ \phi(m) = \dfrac{cy}{\sqrt{x^2+y^2}}$:

substituting these in (2), we find

$$\left(x - \frac{cx}{\sqrt{x^2+y^2}}\right)^2 + \left(y - \frac{cy}{\sqrt{x^2+y^2}}\right)^2 + z^2 = a^2,$$

$$\text{or } (\sqrt{x^2+y^2} - c)^2 = a^2 - z^2,$$

for the equation to the required surface.

By making $y = 0$, we obtain the equation to the trace of the surface upon the plane of xz, which is here

$$(x-c)^2 + z^2 = a^2;$$

and the trace itself is a circle, as it obviously ought to be.

Developable Surfaces.

335. Developable Surfaces may be supposed to be formed by the successive intersections of a series of consecutive planes drawn according to a given law.

The equation to a plane contains three constants, but in order that we may have any determinate number of consecutive planes drawn after a given law, two of the constants must obviously be functions of the third:

Let the general equation to a plane therefore be

$$z = ax + by + c;$$

then, taking that particular plane which corresponds to $c = m$, and making $a = f(m)$, $b = F(m)$, we shall have

$$z - m = f(m)\,x + F(m)\,y \ldots\ldots\ldots\ldots (1):$$

and since for each value of m we have only one plane, if we differentiate with respect to m, considering x, y and z as constant, we shall pass to the consecutive plane: but this gives

$$-1 = f'(m)\,x + F'(m)\,y \ldots\ldots\ldots\ldots (2);$$

and these last two equations express the relations existing between the co-ordinates x, y, z of the intersection of a given plane and the consecutive one; whence the elimination of m would give the relation subsisting among the co-ordinates of the surface containing all such intersections, or the equation to the required developable surface: but we cannot obtain from (2) the value of m in terms of x and y, and we can therefore deduce only the partial differential equation.

If m were eliminated, it would be a function of x, y and z, or of x and y, since z is always a function of x and y: we may therefore consider m as a function of x and y, take the partial differential of (1), and eliminate f and F just as if m were known explicitly in functions of x and y:

now, from (1) we have

$$z = xf(m) + yF(m) + m:$$

$$\therefore \frac{dz}{dx} = f(m) + \{xf'(m) + yF'(m) + 1\}\frac{dm}{dx}$$

$$= f_1(m), \text{ by } (2):$$

$$\frac{dz}{dy} = F(m) + \{yF'(m) + xf'(m) + 1\}\frac{dm}{dy}$$

$$= F_1(m), \text{ by } (2):$$

$$\therefore \text{ since } \frac{dz}{dx} = f_1(m) \text{ and } \frac{dz}{dy} = F_1(m),$$

$$\text{we shall have } \frac{dz}{dx} = \phi\left(\frac{dz}{dy}\right) \text{ or } p = \phi(q):$$

therefore, by taking the partial differential coefficients we get

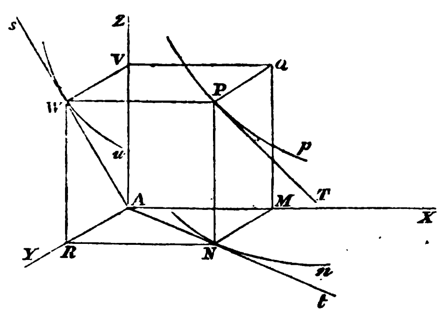
$$r = \phi'(q)\, s \text{ and } s = \phi'(q)\, t:$$

whence $\frac{r}{s} = \frac{s}{t}$, or $rt - s^2 = 0$, is the partial differential equation to all surfaces capable of being developed, or being made to coincide with a plane without rumpling or tearing.

It may readily be proved that the general equations to conical and cylindrical surfaces above found satisfy the one here deduced, and therefore such surfaces are developable.

II. CURVES OF DOUBLE CURVATURE.

336. A Curve of *Double Curvature* being defined by the two equations, $y = f(x)$ and $z = \phi(x)$, which belong to the



two surfaces by whose intersection it is formed; if we suppose AX, AY, AZ to be the three co-ordinate axes, and Pp any arc of it, of which the orthographic projection on the plane of xy is Nn, then, if $AM = x$, $MN = y$ and $NP = z$, $y = f(x)$ will be the equation to the plane curve Nn: similarly, the equation $z = \phi(x)$ belongs to the plane of xz, and by the combination of these two equations may

easily be deduced a third between y and z, which will belong to the curve Ww in the plane of yz, determined as before: and any two of such equations will be sufficient for the determination of the points of the curve.

337. *To find the equations to the Tangent at any point of a curve of double curvature.*

Let the equations to the required tangent be

$$y' = mx' + \mu \text{ and } z' = nx' + \nu:$$

then, when x becomes $x + h$, the corresponding values of y and z will be

$$y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \&c.,$$

$$z + \frac{dz}{dx}h + \frac{d^2z}{dx^2}\frac{h^2}{1.2} + \&c.:$$

and those of y' and z' will similarly be

$$y' + \frac{dy'}{dx'}h + \frac{d^2y'}{dx'^2}\frac{h^2}{1.2} + \&c.,$$

$$z' + \frac{dz'}{dx'}h + \frac{d^2z'}{dx'^2}\frac{h^2}{1.2} + \&c.,$$

$$\text{wherein } \frac{dy'}{dx'} = m, \quad \frac{dz'}{dx'} = n, \quad \frac{d^2y'}{dx'^2} = 0, \quad \frac{d^2z'}{dx'^2} = 0, \ \&c.:$$

whence, making x', y', z' equal to x, y, z respectively in order that the straight line may pass through the proposed point, we shall have the differences between the new values of y, y' and z, z', expressed by

$$\left(\frac{dy}{dx} - m\right)h + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \&c.,$$

$$\left(\frac{dz}{dx} - n\right)h + \frac{d^2z}{dx^2}\frac{h^2}{1.2} + \&c.:$$

now, in order that the distances expressed by these quantities may be the least possible, we must obviously have

$$\frac{dy}{dx} = m \text{ and } \frac{dz}{dx} = n,$$

and the straight lines determined by these conditions will touch the curve more closely than any other: the equations to the required line thus become

$$y' = \frac{dy}{dx} x' + \mu \text{ and } z' = \frac{dz}{dx} x' + \nu:$$

and since it passes through the point x, y, z of the curve, we have likewise

$$y = \frac{dy}{dx} x + \mu \text{ and } z = \frac{dz}{dx} x + \nu:$$

whence, by subtraction, the equations to the rectilineal tangent are

$$y' - y = \frac{dy}{dx}(x' - x) \text{ and } z' - z = \frac{dz}{dx}(x' - x).$$

The same results might have been deduced from considering that the projections of the tangent on the co-ordinate planes are the tangents to the projections of the curve on the same planes.

338. Cor. 1. Hence the tangent may be constructed: for by making $x' = 0$, we shall have

$$y' = y - x\frac{dy}{dx} \text{ and } z' = z - x\frac{dz}{dx},$$

for the co-ordinates of the point in which the tangent meets the plane of zy, and the straight line joining this and the proposed point of the curve will be the tangent.

339. Cor. 2. The length of the tangent intercepted between the point of contact and the plane of zy is equal to

$$\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}$$

when $x'=0$, and is therefore expressed by

$$\sqrt{x^2+x^2\frac{dy^2}{dx^2}+x^2\frac{dz^2}{dx^2}}=x\sqrt{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2};$$

and similarly for the other co-ordinate planes.

Also, the inclination of this line to the plane of zy is

$$\sin^{-1}\frac{x}{x\sqrt{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2}}$$

$$=\sin^{-1}\frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2}}.$$

340. Cor. 3. If s denote the arc of the curve corresponding to the co-ordinates x, y, z, we shall obviously have

$$\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2},$$

which is the differential coefficient of the length of the arc of a curve of double curvature.

Hence, also by the last article $\frac{dx}{ds}$, $\frac{dy}{ds}$ and $\frac{dz}{ds}$ will be the values of the sines of the angles at which the curve is inclined to the co-ordinate planes.

341. *To find the equation to the Normal plane, or the plane which is perpendicular to the direction of a curve of double curvature at any point.*

Let the equation to the required plane be

$$Ax' + By' + Cz' + 1 = 0:$$

then since it passes through the point x, y, z, we shall have

$$A(x' - x) + B(y' - y) + C(z' - z) = 0:$$

also since it is perpendicular to the tangent whose equations are

$$y' - y = \frac{dy}{dx}(x' - x) \text{ and } z' - z = \frac{dz}{dx}(x' - x),$$

we readily obtain

$$\frac{A}{C} = \frac{dx}{dz} \text{ and } \frac{B}{C} = \frac{dy}{dz}:$$

whence the equation to the normal plane becomes

$$\frac{dx}{dz}(x' - x) + \frac{dy}{dz}(y' - y) + (z' - z) = 0,$$

which is sometimes written in the form

$$(x' - x)\,dx + (y' - y)\,dy + (z' - z)\,dz = 0.$$

342. *To find a Surface of given species which shall osculate a given curve of double curvature at any point.*

Let x, y, z be the co-ordinates of the curve at the proposed point, and x', y', z' those of any point of the curve surface: then when x becomes $x + h$, y and z become respectively

$$y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \&c.,$$

$$z + \frac{dz}{dx}h + \frac{d^2z}{dx^2}\frac{h^2}{1.2} + \&c.:$$

similarly x' becoming $x' + h$ causes y' to become

$$y' + \frac{dy'}{dx'}h + \frac{d^2y'}{dx'^2}\frac{h^2}{1.2} + \&c. = y' + k:$$

whence we shall have the new value of z' expressed by

$$z' + \frac{dz'}{dx'}h + \frac{dz'}{dy'}k + \frac{1}{1.2}\left\{\frac{d^2z'}{dx'^2}h^2 + 2\frac{d^2z'}{dx'dy'}hk + \frac{d^2z'}{dy'^2}k^2\right\} + \&c.,$$

which, if for k be put its value above determined, assumes the form

$$z' + Ph + Q\frac{h^2}{1.2} + \&c.:$$

then changing x', y', z' into x, y, z, we find the distance between the curve and surface estimated in the direction of z to be

$$\left(P - \frac{dz}{dx}\right)h + \left(Q - \frac{d^2z}{dx^2}\right)\frac{h^2}{1.2} + \&c.:$$

so that for a contact of the first order we must have $P = \dfrac{dz}{dx}$:

for one of the second order Q must $= \dfrac{d^2z}{dx^2}$ also, and so on.

Ex. To find the equation to the plane osculating a curve of double curvature.

If the equation to the required plane be

$$z' = Ax' + By' + C,$$

we have

$$\frac{dz'}{dx'} = A,\ \frac{dz'}{dy'} = B,\ \frac{d^2z'}{dx'^2} = 0,\ \frac{d^2z'}{dy'^2} = 0,\ \&c.;$$

therefore since the plane passes through the point x, y, z, we shall have the new value of z' expressed by

$$z + \left(A + B\frac{dy}{dx}\right)h + B\frac{d^2y}{dx^2}\frac{h^2}{1.2} + \&c.:$$

also from the first equation we have

$$z' - z = A(x' - x) + B(y' - y),$$

so that to find the constants A and B, we must make

$$A + B\frac{dy}{dx} = \frac{dz}{dx} \text{ and } B\frac{d^2y}{dx^2} = \frac{d^2z}{dx^2}:$$

and these give $B = \dfrac{d^2z}{d^2y}$, $A = \dfrac{dz}{dx} - \dfrac{dy}{dx}\dfrac{d^2z}{d^2y}$:

whence the equation to the plane is

$$z' - z = \left\{\frac{dz}{dx} - \frac{dy}{dx}\frac{d^2z}{d^2y}\right\}(x' - x) + \frac{d^2z}{d^2y}(y' - y),$$

which has with the curve a contact of the second order.

It is manifest that the curvature of the curve lies in the plane thus determined, and may therefore be found from the expression $\gamma = \dfrac{\left\{1 + \left(\dfrac{dz}{dv}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2z}{dv^2}}$, if v and z be the co-ordinates of the curve referred to this plane: and also that this plane is perpendicular to the intersection of two consecutive normal planes at the proposed point.

343. The principles already explained will enable us to ascertain the position and magnitude of a sphere which shall osculate with a curve of double curvature at any point, and thus to find the magnitude of what is called the *spherical* curvature there to distinguish it from the *absolute* curvature just alluded to, and thence the loci of the radii of curvature whether absolute or spherical which will constitute the evolutes of the said curve: and to determine the points of Inflexion of a curve of double curvature, it will generally be sufficient to find the points of that description which belong to the projections of the curve on the co-ordinate planes.

CHAP. XV.

Miscellaneous Theorems and Problems.

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344. *It is required to prove the following formula:* $m^m - \frac{m}{1}(m-1)^m + \frac{m(m-1)}{1.2}(m-2)^m - \&c.$ *continued to* m *terms* $= 1.2.3.\&c.\ m.$

Denoting the coefficients of the terms of an expanded binomial by A, B, C, &c., we have

$$x^m - Ax^{m-1} + Bx^{m-2} - Cx^{m-3} + \&c. = (x-1)^m;$$

whence by differentiation we obtain

$$mx^{m-1} - (m-1)Ax^{m-2} + (m-2)Bx^{m-3} - (m-3)Cm^{m-4} + \&c.$$
$$= m(x-1)^{m-1}:$$

multiplying this result by x we have

$$mx^m - (m-1)Ax^{m-1} + (m-2)Bx^{m-2} - (m-3)Cx^{m-3} + \&c.$$
$$= mx(x-1)^{m-1}:$$

differentiating again and multiplying by x we find

$$m^2x^m - (m-1)^2Ax^{m-1} + (m-2)^2Bx^{m-2} - (m-3)^2Cx^{m-3} + \&c.$$
$$= mx(x-1)^{m-1} + m(m-1)x^2(x-1)^{m-2};$$

a similar process gives

$$m^3x^m - (m-1)^3Ax^{m-1} + (m-2)^3Bx^{m-2} - (m-3)^3Cx^{m-3} + \&c.$$
$$= mx(x-1)^{m-1} + 3m(m-1)x^2(x-1)^{m-2}$$
$$+ m(m-1)(m-2)x^3(x-1)^{m-3}:$$

and repeating this operation n times we shall arrive at the form

$$m^n x^m - (m-1)^n A x^{m-1} + (m-2)^n B x^{m-2} - (m-3)^n C x^{m-3} + \&c.$$
$$= m x (x-1)^{m-1} + P x^2 (x-1)^{m-2} + Q x^3 (x-1)^{m-3} + \&c.$$
$$+ V x^n (x-1)^{m-n}:$$

whence if $n = m$, and therefore $V = m(m-1)(m-2)$ &c. $3.2.1$, and x be supposed $= 1$, we shall obviously have

$$m^m - (m-1)^m A + (m-2)^m B - (m-3)^m C + \&c. = 1.2.3.\&c.\, m.$$

Hence also if n be less or greater than m, we shall have

$$m^n - (m-1)^n A + (m-2)^n B - (m-3)^n C + \&c. = 0 \text{ or } \infty.$$

345. *To investigate the law of the formation of the coefficients of the terms of the Multinomial Theorem by means of the Differential Calculus.*

Let $(a_0 + a_1 x + a_2 x^2 + \&c.)^m = b_0 + b_1 x + b_2 x^2 + \&c.$, which gives immediately

$$m \log (a_0 + a_1 x + a_2 x^2 + \&c.) = \log (b_0 + b_1 x + b_2 x^2 + \&c.):$$

whence differentiating both sides we obtain

$$\frac{m(a_1 + 2a_2 x + 3a_3 x^2 + \&c.)}{a_0 + a_1 x + a_2 x^2 + \&c.} = \frac{b_1 + 2b_2 x + 3b_3 x^2 + \&c.}{b_0 + b_1 x + b_2 x^2 + \&c.}:$$

and this being multiplied out and the coefficients of the same powers of x equated gives

$$b_1 = m a_1 \frac{b_0}{a_0}, \text{ where } b_0 \text{ obviously } = a_0{}^m;$$

$$b_2 = (2m-0)\, a_2 \frac{b_0}{2a_0} + (m-1)\, a_1 \frac{b_1}{2a_0};$$

$$b_3 = (3m-0)\, a_3 \frac{b_0}{3a_0} + (2m-1)\, a_2 \frac{b_1}{3a_0} + (m-2)\, a_1 \frac{b_2}{3a_0};$$

&c. = &c...

and thus the law is manifest; and this agrees with what has been said in (218) of the *Algebra*.

If $a_2 = a_3 = \&c. = 0$, we shall readily find

$$b_1 = m a_1 \frac{b_0}{a_0} = m a_1 a_0^{m-1};$$

$$b_2 = (m-1)\, a_1 \frac{b_1}{2a_0} = \frac{m(m-1)}{1.2} a_1^2 a_0^{m-2};$$

and so on as in the *Binomial Theorem*.

346. *It is required to decompose* $\frac{1}{x^m - 1}$ *into a series of fractions having simple or quadratic denominators.*

First let m be odd, then in (291) of the *Trigonometry*, we have seen that

$$x^m - 1 = (x-1)\left(x^2 - 2\cos\frac{2\pi}{m}x + 1\right)\left(x^2 - 2\cos\frac{4\pi}{m}x + 1\right) \&c.$$

to $\frac{1}{2}(m+1)$ factors: whence we get

$$\log(x^m - 1) = \log(x-1) + \log\left(x^2 - 2\cos\frac{2\pi}{m}x + 1\right)$$

$$+ \log\left(x^2 - 2\cos\frac{4\pi}{m}x + 1\right) + \&c. \text{ to } \tfrac{1}{2}(m+1) \text{ terms}:$$

differentiating both sides we obtain

$$\frac{m x^{m-1}}{x^m - 1} = \frac{1}{x-1} + \frac{2x - 2\cos\frac{2\pi}{m}}{x^2 - 2\cos\frac{2\pi}{m}x + 1}$$

$$+ \frac{2x - 2\cos\frac{4\pi}{m}}{x^2 - 2\cos\frac{4\pi}{m}x + 1} + \&c. \text{ to } \tfrac{1}{2}(m+1) \text{ terms}:$$

$$\therefore \frac{mx^m}{x^m-1} = \frac{x}{x-1} + \frac{2x^2 - 2x\cos\frac{2\pi}{m}}{x^2 - 2x\cos\frac{2\pi}{m} + 1}$$

$$+ \frac{2x^2 - 2x\cos\frac{4\pi}{m}}{x^2 - 2x\cos\frac{4\pi}{m} + 1} + \text{\&c. to } \tfrac{1}{2}(m+1) \text{ terms}:$$

and $m = 1 + 2 + 2 + \text{\&c.}$ to $\frac{1}{2}(m+1)$ terms: therefore by the subtraction of the corresponding terms we obtain

$$\frac{m}{x^m-1} = \frac{1}{x-1} + \frac{2x\cos\frac{2\pi}{m} - 2}{x^2 - 2x\cos\frac{2\pi}{m} + 1}$$

$$+ \frac{2x\cos\frac{4\pi}{m} - 2}{x^2 - 2x\cos\frac{4\pi}{m} + 1} + \text{\&c. to } \tfrac{1}{2}(m+1) \text{ terms},$$

both sides of which being divided by m give what was required.

If m be even, a similar process may be used, and each of the fractions with quadratic denominators may again be decomposed into others with denominators involving only the simple power of x.

Similarly of the fraction $\dfrac{1}{x^{2m} \pm 2x^m \cos a + 1}$, &c.

347. *To find the sum of* n *terms of the series*

$$1^2 \sin\theta + 3^2 \sin 3\theta + 5^2 \sin 5\theta + \text{\&c.}$$

It has been proved in (310) of the *Trigonometry*, that

$$\sin\theta + \sin 3\theta + \sin 5\theta + \text{\&c. to } m \text{ terms} = \frac{\sin^2 m\theta}{\sin\theta}:$$

whence differentiating twice in succession with respect to θ we have

$$\cos\theta + 3\cos 3\theta + 5\cos 5\theta + \&c. = \frac{m\sin 2m\theta \sin\theta - \sin^2 m\theta \cos\theta}{\sin^2\theta}:$$

and thence, after changing the signs, the value of

$$1^2 \sin\theta + 3^2 \sin 3\theta + 5^2 \cos 5\theta + \&c. \text{ to } m \text{ terms}$$

is readily found. Similarly of a variety of others.

348. *It is required to find the form of the function* f, *so that* $f(x+h)+f(x-h)=f(x)f(h)$.

Let $u=f(x)$; then we have by *Taylor's* Theorem

$$f(x+h) = u + \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \&c.,$$

$$f(x-h) = u - \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1.2} - \&c.:$$

whence by addition and division by u, we get

$$\frac{f(x+h)+f(x-h)}{u} = f(h)$$

$$= 2\left\{1 + \frac{d^2u}{udx^2}\frac{h^2}{1.2} + \frac{d^4u}{udx^4}\frac{h^4}{1.2.3.4} + \&c.\right\}:$$

but since $f(h)$ is entirely independent of x, it follows that the coefficients of the different powers of h must here be constant quantities: wherefore if $\frac{d^2u}{udx^2} = a$ or $\frac{d^2u}{dx^2} = au$, we shall readily obtain

$$\frac{d^4u}{dx^4} = \frac{d}{dx}\frac{d^3u}{dx^3} = \frac{d}{dx}a\frac{du}{dx} = a\frac{d^2u}{dx^2} = a^2u:$$

$$\frac{d^6u}{dx^6} = \frac{d}{dx}\frac{d^5u}{dx^5} = \frac{d}{dx}a^2\frac{du}{dx} = a^2\frac{d^2u}{dx^2} = a^3u, \text{ and so on:}$$

wherefore we have now

$$f(h) = 2\left\{1 + \frac{ah^2}{1.2} + \frac{a^2h^4}{1.2.3.4} + \&c.\right\}$$

$$= 2\left\{1 - \frac{c^2h^2}{1.2} + \frac{c^4h^4}{1.2.3.4} - \&c.\right\},$$

if $-c^2$ be put in the place of a: that is, $f(h) = 2\cos ch$, and thus the form of f is determined: and it may readily be verified for

$$2\cos(x+h) + 2\cos(x-h) = 4\cos x \cos h = (2\cos x)(2\cos h).$$

349. *To find the particular value of the fraction*

$$u = \frac{(F_1x)^{\frac{m}{n}} + (F_2x)^{\frac{m}{n}} + \&c.}{(f_1x)^{\frac{m}{n}} + (f_2x)^{\frac{m}{n}} + \&c.},$$

when such a value a *is assigned to* x *as causes each of the terms to vanish.*

By *Taylor's* Theorem, the numerator and denominator corresponding to $x+h$ are

$$\left\{F_1(x) + \frac{dF_1(x)}{dx}h + \frac{d^2F_1(x)}{dx^2}\frac{h^2}{1.2} + \&c.\right\}^{\frac{m}{n}}$$

$$+\left\{F_2(x) + \frac{dF_2(x)}{dx}h + \frac{d^2F_2(x)}{dx^2}\frac{h^2}{1.2} + \&c.\right\}^{\frac{m}{n}}$$

$+\&c$, and

$$\left\{f_1(x) + \frac{df_1(x)}{dx}h + \frac{d^2f_1(x)}{dx^2}\frac{h^2}{1.2} + \&c.\right\}^{\frac{m}{n}}$$

$$+\left\{f_2(x) + \frac{df_2(x)}{dx}h + \frac{d^2f_2(x)}{dx^2}\frac{h^2}{1.2} + \&c.\right\}^{\frac{m}{n}}$$

$+\&c.$:

but when $x=a$, we have supposed each of the terms in the numerator and denominator of the proposed fraction to

vanish: whence, dividing by $h^{\frac{m}{n}}$, it is clear that the true value will be obtained by differentiating each of the quantities without regard to their exponents: and if this result $=\frac{0}{0}$, the same steps must obviously be repeated, and so on.

Ex. Let $u=\dfrac{\sqrt[3]{a^2-x^2}+\sqrt[3]{(a-x)^2}}{\sqrt[3]{a-x}-\sqrt[3]{a^3-x^3}}$, which $=\frac{0}{0}$ when $x=a$; then, differentiating each quantity under the radical sign, we have the particular value of u

$$=\frac{\sqrt[3]{-2x}+\sqrt[3]{-2(a-x)}}{\sqrt[3]{-1}-\sqrt[3]{-3x^2}}=\frac{\sqrt[3]{2a}}{1-\sqrt[3]{3a^2}};$$

which may be verified by the ordinary process.

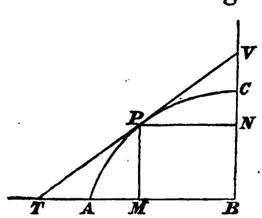
350. Cor. The same method is applicable in all cases where the indices are the same, and each term vanishes of itself: but if two or more separate terms destroy each other, the same process may be adopted with the rest, no regard whatever being paid to these as in

$$u=\frac{\sqrt{x^2+a}-\sqrt{a^2+x}+\sqrt{a^2-x^2}}{\sqrt{a^3-x^3}},$$

whose value is $\dfrac{\sqrt{-2x}}{\sqrt{-3x^2}}=\sqrt{\dfrac{2}{3a}}$ when $x=a$.

351. *To find the shortest straight line that can be drawn through a given point between two other straight lines at right angles to each other.*

Let the two given indefinite straight lines be BT and



BV, and *P* the given point: then, if $MB = a$, $MP = b$ and the angle $PTM = \theta$, the equation to TV will be

$$y' - b = -\tan\theta\,(x' - a);$$

$$\text{whence, if } y' = 0,\quad BT = \frac{a\tan\theta + b}{\tan\theta}:$$

$$\text{and if } x' = 0,\quad BV = a\tan\theta + b:$$

$$\therefore\ TV^2 = \frac{(a\tan\theta + b^2)(1 + \tan^2\theta)}{\tan^2\theta}:$$

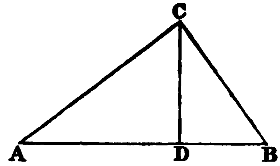
but if $TM = x$, then $\tan\theta = \dfrac{b}{x}$, and we must have

$$u = (a + x)^2\left(1 + \frac{b^2}{x^2}\right) \text{ a } \textit{minimum}:$$

whence $\dfrac{du}{dx} = 0$ gives $x = -a$ and $x = \sqrt[3]{ab^2}$, the former of which by the usual criterion belongs to a maximum, and the latter to a minimum which $= (a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$.

352. *Of all triangles having a given base and vertical angle, to find that whose area is the greatest.*

Let the given base $AB = c$ and the given vertical angle



$ACB = a$; then, putting $AD = x$, we have $DB = c - x$: whence the area of the $\triangle = \frac{1}{2}AB\,.\,CD \propto CD = u$:

but $\tan a = \tan(ACD + BCD)$

$$= \frac{\tan ACD + \tan BCD}{1 - \tan ACD\tan BCD} = \frac{cu}{u^2 - x(c - x)}:$$

$$\therefore\ u^2 \tan a - cu = (cx - x^2) \tan a,$$

in which u is to be a *maximum*: and this gives

$$(2u \tan a - c) \frac{du}{dx} = (c - 2x) \tan a:$$

therefore, from $\dfrac{du}{dx} = 0$ we obtain $x = \frac{1}{2}c$, so that the required triangle is isosceles and its area $= \dfrac{c^2}{4} (\cot a + \operatorname{cosec} a)$.

353. *To find the radius of a Circle such that the segment of it formed by an arc of given length may be the greatest possible.*

Let the given length of the arc $= a$ and the radius of the circle $= x$: then will $\dfrac{a}{x} =$ the angle of the sector: whence we must have twice the area of the segment, which is

$$u = ax - x^2 \sin \frac{a}{x} \cos \frac{a}{x} = ax - \frac{1}{2} x^2 \sin \frac{2a}{x}, \text{ a maximum:}$$

$$\therefore\ \frac{du}{dx} = a - x \sin \frac{2a}{x} + a \cos \frac{2a}{x} = 0,$$

which will be satisfied if $x = \dfrac{2a}{\pi}$ and $x = \infty$: therefore the area of the greatest sector $= \dfrac{2a^2}{\pi}$, and that of the least $= 0$.

354. *To find the greatest Quadrilateral Figure that can be formed by four given straight lines taken in a given order.*

Let a, a', c, c' be the sides meeting in the angles A and C respectively, a being adjacent to c and a' to c': then, if θ and ϕ denote the said angles, we shall have twice the area of the figure expressed by

$$u = aa' \sin \theta + cc' \sin \phi, \text{ a maximum:}$$

but the diagonal of the figure subtending the angles at A and C being the same for both, we shall have an equation of condition

$$a^2 + a'^2 - 2aa' \cos\theta = c^2 + c'^2 - 2cc' \cos\phi:$$

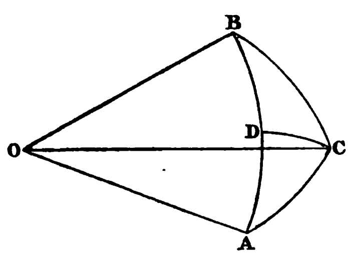
whence, eliminating $\frac{d\phi}{d\theta}$ between these two equations, we find $\phi = \pi - \theta$, or the quadrilateral is inscribable in a circle, and its area therefore

$$= \sqrt{(s-a)(s-a')(s-c)(s-c')}, \text{ if } 2s = a + a' + c + c'.$$

From this it appears, that the area of the greatest quadrilateral is independent of the order in which the lines occur, though the angles are not.

355. *Of all Spherical Triangles upon the same base and having equal perpendiculars, to find that which has the greatest vertical angle.*

Let ABC be the spherical triangle, having the side



$AB = c$ and the perpendicular $CD = k$, and let $AD = \theta$: then *Napier's* Rules give

$$\tan ACD = \frac{\tan\theta}{\sin k} \text{ and } \tan BCD = \frac{\tan(c-\theta)}{\sin k}:$$

whence we have

$$\tan C = \tan(ACD + BCD) = \frac{\sin k\,\{\tan\theta + \tan(c-\theta)\}}{\sin^2 k - \tan\theta\tan(c-\theta)}$$

$$= \frac{\sin c \sin k}{\cos c - \cos^2 k \cos\theta \cos(c-\theta)}, \textit{ a maximum:}$$

from which we have $u = \cos\theta \cos(c-\theta) =$ *a maximum:*

$$\therefore \frac{du}{dx} = -\sin\theta\cos(c-\theta) + \cos\theta\sin(c-\theta) = 0,$$

which gives $\tan\theta = \tan(c-\theta)$, or $\theta = c - \theta$;

whence $\theta = \frac{1}{2}c$, or the triangle is isosceles: and this conclusion being entirely independent of the magnitude of the radius of the sphere, must hold good when that radius becomes infinite, and therefore in a plane triangle.

356. *To determine the greatest Curvilineal Figure of a given species which can be inscribed in, and the least which can be described about another given curvilineal figure.*

Let $y = F(x)$ be the equation of the given figure, and $y_{,} = f(\alpha, \beta, x_{,})$ that of the figure of the proposed species, the individual figure being defined by the values of α and β:

then, since the figures touch each other when $x_{,} = x$, we must obviously have

$$F(x) = f(\alpha, \beta, x) \text{ and } F'(x) = f'(\alpha, \beta, x):$$

from which, if x be eliminated, there results the form

$$\psi(\alpha, \beta) = 0:$$

but the area represented by $\phi(\alpha, \beta)$ being a maximum or a minimum, we must have $\phi'(\alpha, \beta) = 0$:

whence the two equations

$$\psi(\alpha, \beta) = 0 \text{ and } \phi'(\alpha, \beta) = 0,$$

give the values of α and β corresponding to a maximum or a minimum: which may be verified by the ordinary method.

Ex. 1. To inscribe the greatest Ellipse in a given triangle.

Let BAC be the proposed triangle, and let $AD=a$ be drawn to bisect the side BC in D: then, if D be the origin of co-ordinates and DA, BDC their axes, we shall have

$y=m(a-x)$ for the equation to the straight line BA;

$y_{,}=\dfrac{\beta}{\alpha}(2\alpha x_{,}-x_{,}^2)^{\frac{1}{2}}$ the equation to an ellipse whose magnitude is dependent upon the quantities α, β to be determined:

$\therefore$ by reason of their contact we must have the two equations

$$m(a-x)=\frac{\beta}{\alpha}(2\alpha x-x^2)^{\frac{1}{2}},$$

$$\text{and}\quad -m=\frac{\beta}{\alpha}\,\frac{\alpha-x}{(2\alpha x-x^2)^{\frac{1}{2}}};$$

$$\text{whence we get}\quad x-a=\frac{2\alpha x-x^2}{\alpha-x},\ \text{or}\ x=\frac{a\alpha}{a-\alpha}$$

also, by multiplying together these equations, we obtain

$$\frac{\beta^2}{\alpha^2}=\frac{m^2(x-a)}{\alpha-x}=\frac{m^2(a^2-2a\alpha)}{\alpha^2},$$

which gives immediately $\beta^2=m^2(a^2-2a\alpha)$,

from which x has been eliminated;

$$\therefore\ \frac{d\beta}{d\alpha}=-\frac{m^2a}{\beta}:$$

but $u=\alpha\beta=$ *a maximum*, from which we obtain

$$\frac{d\beta}{d\alpha}=-\frac{\beta}{\alpha};$$

$$\therefore\ m^2a\alpha=\beta^2=m^2(a^2-2a\alpha),\ \text{or}\ \alpha=\tfrac{1}{3}a\ \text{and}\ \therefore\ \beta=\frac{ma}{\sqrt{3}}:$$

and the area of the greatest ellipse will therefore be

$$= \pi \alpha \beta \sin \gamma = \frac{m \pi a^2 \sin \gamma}{3\sqrt{3}},$$

if γ represent the angle at which the co-ordinate axes are inclined to each other.

It appears from this that all the sides of the triangle are bisected by the points of contact, if any one be so.

Ex. 2. To find the least Parabola that can circumscribe a given circle.

Let $y = \sqrt{2ax - x^2}$ be the equation to the circle whose radius is a; α and β the *latus rectum* and length of the axis of a circumscribed parabola, so that its equation is

$$y_{,} = \sqrt{\alpha(\beta - x_{,})}:$$

then, in consequence of their contact, we must have

$$\sqrt{2ax - x^2} = \sqrt{\alpha(\beta - x)} \text{ and } \frac{a - x}{\sqrt{2ax - x^2}} = -\frac{1}{2}\frac{\alpha}{\sqrt{\alpha\beta - \alpha x}},$$

from which is found $x = a + \frac{1}{2}\alpha$:

whence, by substitution, we obtain

$$\beta = \frac{a^2}{\alpha} + a + \frac{1}{4}\alpha, \text{ and } \therefore \frac{d\beta}{d\alpha} = \frac{1}{4} - \frac{a^2}{\alpha^2}:$$

but the area of the parabola $= \frac{2}{3}$ of its circumscribed rectangle $= \frac{4}{3}\beta\sqrt{\alpha\beta}$, and thence $u = \alpha\beta^3 = a$ *maximum*, which gives $\dfrac{d\beta}{d\alpha} = -\dfrac{\beta}{3\alpha}$:

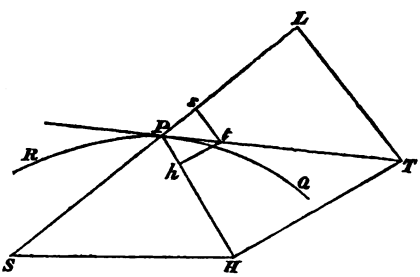
$$\therefore \frac{a^2}{\alpha^2} - \frac{1}{4} = \frac{\beta}{3\alpha} = \frac{a^2}{3\alpha^2} + \frac{a}{3\alpha} + \frac{1}{12}:$$

whence $a^2 + \alpha a - 2a^2 = 0$, whose roots are a and $-2a$, the latter of which is useless;

that is, $\alpha = a$, or the *latus rectum* of the parabola is equal to the radius of the circle: also, the length of the axis of the least parabola $= \frac{9}{4}a$, and its area $= \frac{9}{2}a^2$.

357. *To draw a Tangent to a curve defined by an equation between two straight lines drawn to two given points.*

Let S and H be the two given points, $SP = x$ and $HP = y$ the straight lines whose relation is given, such that



$f(x, y) = 0$: suppose PT a tangent to the curve at P; draw HT perpendicular to HP and TL perpendicular to SP produced, and parallel to these draw ht and ts near to P:

$$\text{then } \frac{PL}{PH} = \frac{Ps}{Ph}, \text{ or } PL = PH\frac{Ps}{Ph}:$$

and since this always holds good, we shall obviously have

$$PL = PH \text{ limit of } \frac{Ps}{Ph} = -PH\frac{dx}{dy} = -\frac{y\,dx}{dy},$$

which is therefore known in terms of x or y; hence conversely, if PL be made equal to this magnitude, the point T in which the perpendiculars from L and H to SL and HP respectively, intersect, is a point in the required tangent, which may therefore be constructed.

Ex. 1. Let the relation between the two straight lines be expressed by the equation $x^m + y^n = a^m$.

Here we have $mx^{m-1} + ny^{n-1}\dfrac{dy}{dx} = 0$,

$$\therefore\ PL = -\frac{y\,dx}{dy} = \frac{ny^n}{mx^{m-1}} = \frac{n(a^m - x^m)}{mx^{m-1}},$$

which determines the position of the point L for any point of the curve, and thus T being found, the tangent may be drawn.

If $m = n = 1$ or $x + y = a$, the curve is an Ellipse: also $PL = HP$, and therefore the tangent PT bisects the angle HPL, as is well known.

Ex. 2. Let the equation be $x^m y^n = a^{m+n}$: then we have

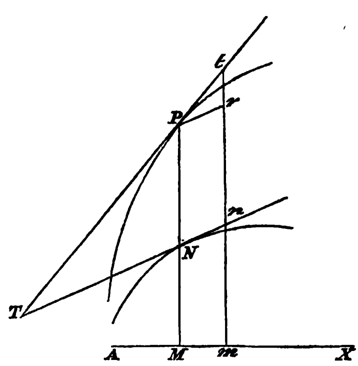
$$mx^{m-1}y^n + nx^m y^{n-1}\frac{dy}{dx} = 0:$$

$\therefore\ PL = \dfrac{nx}{m}$, and the tangent may be drawn.

If $m = n = 1$ we have $xy = a^2$, for the Lemniscata of *Bernoulli*: also in this case $PL = x = SP$.

358. *To draw a Tangent to a curve defined by an equation between a rectilineal ordinate and a curvilineal abscissa.*

Let AN here considered as the abscissa $= s$, NP the ordi-



nate $=y$, and let the equation be $f(y, s)=0$: then if we suppose tangents to be drawn at N and P meeting in T, NT may be called the subtangent of the curve AP: drawing also mnt parallel to NP and Pr parallel to TNn, we shall have $\frac{TN}{NP}=\frac{Pr}{rt}$, which is true, however small they may be:

$$\therefore\ TN = NP \text{ limit of } \frac{Nn}{rt} = \frac{y\,ds}{dy},$$

which being known in terms of y and s, the tangent may be constructed.

Ex. Let the equation be $y^m = a^{m-1}s$: then we have $TN=\frac{my^m}{a^{m-1}}=ms$, so that PT will be a tangent at P.

If $m=1$ and $a=1$, the curve AP becomes a Cycloid, and we have $TN=y=NP=AN$; from which it also appears that the locus of the point T is the Involute of the generating circle.

359. *To draw a common Tangent to two curves defined by given equations.*

Let $y_1=F(x_1)$ and $y_2=f(x_2)$ be the equations to the two curves, their co-ordinates being measured from the same point: then the equations to their tangents are

$$y'-y_1=\frac{dy_1}{dx_1}(x'-x_1),$$

$$y''-y_2=\frac{dy_2}{dx_2}(x''-x_2):$$

and in order that these two tangents may coincide, it is obvious that they must meet the axis of x in the same point and make the same angle with it: whence if $y'=0$, $y''=0$, and therefore

$$x'\frac{dy_1}{dx_1}=x_1\frac{dy_1}{dx_1}-y_1 \text{ and } x''\frac{dy_2}{dx_2}=x_2\frac{dy_2}{dx_2}-y_2:$$

by making $x' = x''$ and $\frac{dy_1}{dx_1} = \frac{dy_2}{dx_2}$ we shall obtain

$$x_1 \frac{dy_1}{dx_1} - y_1 = x_2 \frac{dy_1}{dx_1} - y_2,$$

$$\text{or } (x_1 - x_2) \frac{dF(x_1)}{dx_1} = y_1 - y_2 = F(x_1) - f(x_2):$$

which therefore establishes a relation between x_1 and x_2: and by means of this combined with $\frac{dy_1}{dx_1} = \frac{dy_2}{dx_2}$, the values of x_1 and x_2 may be found, and the common tangent drawn.

Ex. Let the curves be a circle and parabola on the same axis.

Here the equations from the centre of the circle are

$$y_1^2 = a^2 - x_1^2 \text{ and } y_2^2 = 4a(x_2 - \beta);$$

and their respective tangents are defined by the equations

$$y_1 y' + x_1 x' = a^2 \text{ and } y_2 y'' = 2ax'' + 2ax_2 - 4a\beta:$$

whence making $y'' = y' = 0$, and $x'' = x'$ we obtain

$$\frac{a^2}{x_1} = 2\beta - x_2 \text{ or } x_2 = 2\beta - \frac{a^2}{x_1}:$$

$$\text{also } \frac{dy_1}{dx_1} = \frac{dy_2}{dx_2} \text{ gives } \frac{x_1}{y_1} = \frac{2a}{y_2} = \frac{2a}{\sqrt{4a(x_2 - \beta)}}:$$

$$\text{whence } x_2 = \beta + \frac{a(a^2 - x_1^2)}{x_1^2}:$$

and thence we shall have, after proper reduction, the equation

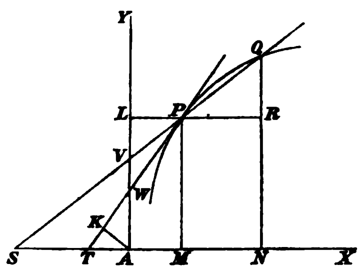
$$(a + \beta) x_1^2 - a^2 x_1 - a^2 a = 0,$$

for determining the values of x_1, and thence those of x_2.

The two roots of the equation here deduced shew that the curves may have two common tangents, one when the contacts are on the same side of the axis of x, and the other when on different sides.

360. *To find the Locus of the intersections of the tangent with the perpendicular drawn upon it from the origin in any curve.*

Let P be any point in the curve whose co-ordinates are x, y: PT a tangent there, and AK a perpendicular upon it from the origin:



then the general equation to the tangent PT is

$$y' - y = \frac{dy}{dx}(x' - x):$$

whence if $x_{,}$, $y_{,}$ be the co-ordinates of any point of the line AK, its equation will be $y_{,} = -\frac{dx}{dy}x_{,}$ by (142): but at the point of intersection of AK and PT, we must have $x' = x_{,}$ and $y' = y_{,}$, so that there result the two equations

$$y_{,} - y = \frac{dy}{dx}(x_{,} - x) \text{ and } y_{,} + x_{,}\frac{dx}{dy} = 0,$$

which combined with the equation to the proposed curve $y = f(x)$ will be sufficient to eliminate x and y, and thus to find the relation between $x_{,}$ and $y_{,}$ which will define the locus of the point K: also, we have

$$\frac{y_{,} - y}{x_{,} - x} = \frac{dy}{dx} = -\frac{x_{,}}{y_{,}},$$

and thence we deduce immediately the equations

$$y_{,}^{2}+x_{,}^{2}=xx_{,}+yy_{,} \text{ and } \frac{dy}{dx}=-\frac{x_{,}}{y_{,}},$$

for the equations of condition.

Ex. 1. To find the curve traced out by the intersections of the tangent to a parabola with the perpendicular upon it from the origin.

The equation to the tangent here is easily found to be

$$y'=\frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}(x'+x):$$

also that of the perpendicular upon it is immediately had

$$y_{,}=-\frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}}}x_{,}:$$

therefore belonging to their common intersection where $x'=x_{,}$ and $y'=y_{,}$, we have the two equations

$$y_{,}=\frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}(x_{,}+x) \text{ and } y_{,}+\frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}}}x_{,}=0:$$

but since $\frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}}}=-\frac{y_{,}}{x_{,}}$ we find $x=\frac{ay_{,}^{2}}{x_{,}^{2}}$; and thence the first of these equations becomes

$$y_{,}=-\frac{x_{,}}{y_{,}}\left(x_{,}+\frac{ay_{,}^{2}}{x_{,}^{2}}\right), \text{ or } y_{,}^{2}=-\frac{x_{,}^{3}}{a+x_{,}},$$

which is the equation to the Cissoid of *Diocles*, the change in the sign of $x_{,}$ merely shewing that its abscissa is measured in a contrary direction to that of the parabola.

Ex. 2. If the proposed curve be an ellipse referred to its centre, we find the equation of the tangent to be

$$y'=\frac{b}{a}\frac{a^{2}-xx'}{\sqrt{a^{2}-x^{2}}};$$

and that of the perpendicular upon it from the centre to be

$$y_{,}=\frac{a}{b}\,\frac{x_{,}\sqrt{a^2-x^2}}{x}:$$

hence making $x'=x_{,}$ and $y'=y_{,}$ and eliminating x from the equations

$$y_{,}=\frac{b}{a}\,\frac{a^2-xx_{,}}{\sqrt{a^2-x^2}}\text{ and } y_{,}=\frac{a}{b}\,\frac{x_{,}\sqrt{a^2-x^2}}{x},$$

we arrive at

$$(x_{,}^2+y_{,}^2)^2=a^2x_{,}^2+b^2y_{,}^2,$$

for the equation of the locus required.

361. Cor. 1. The equation to the locus by the perpendicular from any other point upon the tangent may be found by transferring the origin of co-ordinates to the point proposed and proceeding as above.

362. Cor. 2. Hence, of two equal curves of the same species having at first their vertices coincident, and their axes in the same straight line, if one be made to roll upon the other at rest, the locus of the vertex of the rolling curve will be similar to the locus of K above found, since K always bisects the line joining the vertices of the two figures. Thus in the case of two parabolas, the locus is the Cissoid of *Diocles:* and for two equal circles, it will be an Epicycloid which is the only one that can be expressed by an algebraical equation, and is commonly called the *Cardioide*.

363. *To find the Locus of the intersections of two tangents drawn to a given curve after some determinate law.*

Let α, β and α', β' denote any two corresponding points in the curve proposed; then the equations to the tangents at those points are

$$y'-\beta=\frac{d\beta}{d\alpha}(x'-\alpha),$$

$$y''-\beta'=\frac{d\beta'}{d\alpha'}(x''-\alpha'):$$

but at the point of their intersection $x'' = x'$ and $y'' = y'$, whence we get

$$\beta + \frac{d\beta}{d\alpha}(x' - \alpha) = \beta' + \frac{d\beta'}{d\alpha'}(x' - \alpha'):$$

now $\frac{d\beta}{d\alpha}$ and $\frac{d\beta'}{d\alpha'}$ being connected by some determinate law, either of them may be regarded as a function of the other as $\frac{d\beta'}{d\alpha'} = f\frac{d\beta}{d\alpha}$: and since β and β' are the same functions of α and α' respectively, it is obvious that the three equations

$$y' - \beta = \frac{d\beta}{d\alpha}(x' - \alpha),$$

$$\beta + \frac{d\beta}{d\alpha}(x' - \alpha) = \beta' + \frac{d\beta'}{d\alpha'}(x' - \alpha'),$$

$$\text{and } \frac{d\beta'}{d\alpha'} = f\left(\frac{d\beta}{d\alpha}\right),$$

will be sufficient to eliminate both α and α', and thus to deduce the relation between x' and y' considered as the co-ordinates of the required locus.

Ex. Let the curve be a parabola, and the two touching lines be at such inclinations to the axis of x that the sum of their cotangents is constant.

Here we have for the equations to the two tangents

$$y'\beta = 2a(x' + \alpha),$$

$$y''\beta' = 2a(x'' + \alpha'):$$

whence if $x'' = x'$ and $y'' = y'$, we get by subtraction

$$y'(\beta - \beta') = 2a(\alpha - \alpha'):$$

but $\dfrac{d\alpha}{d\beta} + \dfrac{d\alpha'}{d\beta'} = \dfrac{\beta + \beta'}{2a} = m$, by hypothesis, or $\beta + \beta' = 2ma$: whence the three characteristic equations now become

$$y'\beta = 2a(x' + \alpha),$$
$$y'(\beta - \beta') = 2a(\alpha - \alpha'),$$
$$\beta + \beta' = 2ma;$$

from which if α and α' and therefore β and β' be eliminated, the relation between x' and y' will be obtained: thus, we have readily

$$y'(\beta^2 - \beta'^2) = 4ma^2(\alpha - \alpha'):$$

that is,

$$y'(4a\alpha - 4a\alpha') = 4ma^2(\alpha - \alpha'):$$

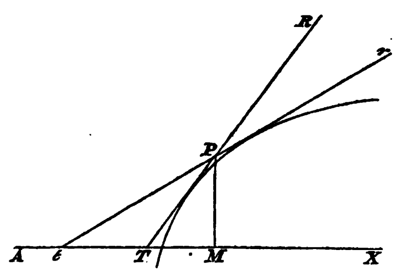
whence by equal division is found

$$y' = ma,$$

which is the equation of a straight line parallel to the axis of x, and whose distance from it $= ma$.

364. *To determine the Nature of a curve which shall always touch any number of straight lines drawn after a given law.*

Let AX be the axis of the required curve, A being the



origin, and suppose TPR to be one of the straight lines drawn according to the proposed law, and let it be intersected by another line tPr of the same description in P: x', y' the co-ordinates of the tangent TPR and x, y those of the required curve, the equation of the former being

$$y' = Px' + Q,$$

wherein P and Q involve only constant quantities, and the variable parameter a which indicates the law according to which the lines are drawn:

now if a be supposed to become $a+h$, then will

$$P \text{ become } P + \frac{dP}{da}h + \frac{d^2P}{da^2}\frac{h^2}{1.2} + \&c.,$$

$$Q \ldots\ldots\ldots Q + \frac{dQ}{da}h + \frac{d^2Q}{da^2}\frac{h^2}{1.2} + \&c.,$$

so that the equation to the tangent tPr assumes the form

$$y'' = (P + \frac{dP}{da}h + \&c.)x' + Q + \frac{dQ}{da}h + \&c.:$$

hence at the point P where the co-ordinates of these two right lines are the same, we must obviously have

$$Px' + Q = (P + \frac{dP}{da}h + \&c.)\,x' + Q + \frac{dQ}{da}h + \&c.,$$

$$\text{or } 0 = \left(\frac{dP}{da}h + \&c.\right)x' + \frac{dQ}{da}h + \&c.:$$

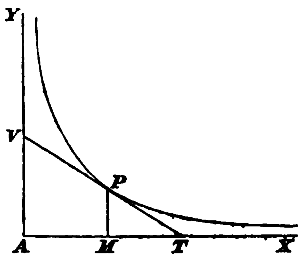
but in the limit, when the tangents become consecutive, the position of the point P is in the touching curve, h becomes $=0$ and $x'=x$; whence we obtain after dividing by h

$$\frac{dP}{da}x + \frac{dQ}{da} = 0:$$

which combined with the equation $y = Px + Q$ will be sufficient for the elimination of the variable parameter a, and the resulting equation being independent of it will necessarily belong to every line answering the proposed conditions, and thus the required curve will be found.

Ex. 1. Given the area of a right-angled triangle, to find the curve to which the hypothenuse is always a tangent.

Let the given area $=2a^2$, $AM=x$, $MP=y$ and $AT=a$



the parameter, by the variation of which the tangent TPV assumes its different positions: then

$$AV=\frac{4a^2}{a}, \text{ and } y=(a-x)\tan ATV=(a-x)\frac{4a^2}{a^2}:$$

$$\therefore\ a^2y=4a^2(a-x):$$

which, differentiated with respect to a alone, gives

$$2ay=4a^2 \text{ or } a=\frac{2a^2}{y}:$$

whence by substitution in the equation above we obtain

$$y=\left(\frac{2a^2}{y}-x\right)\frac{y^2}{a^2}, \text{ or } xy=a^2,$$

the equation to an hyperbola between the asymptotes; and it may be observed that this result agrees entirely with what was said in Ex. 3. of (138).

Ex. 2. Two straight lines are drawn perpendicular to a given straight line at given points, and their rectangle is invariable: it is required to find the curve to which the line joining their extremities is always a tangent.

Let $AB=2a$ be the given line, and AT, BV be two perpendiculars to it at A and B: $AM=x$, $MP=y$, and let $AT\times BV=b^2$: also, let $y'=Px'+Q$ be the equation to TV, and $a=AT$ the variable parameter which decides the posi-

tion of each particular line: then, if $x'=0$, we have $Q=a$; and if $x'=2a$, we get

$$\frac{b^2}{a}=2aP+a, \text{ or } P=\frac{b^2-a^2}{2aa}:$$

whence if x' be changed into x and y' into y, the equation is

$$y=\frac{b^2-a^2}{2aa}x+a, \text{ or } 2aay=(b^2-a^2)x+2aa^2:$$

therefore, differentiating with respect to a, we find

$$2ay=4aa-2ax, \text{ or } a=\frac{ay}{2a-x}:$$

whence the equation above found becomes by substitution

$$\frac{2a^2y^2}{2a-x}=\left\{b^2-\frac{a^2y^2}{(2a-x)^2}\right\}x+\frac{2a^3y^2}{(2a-x)^2},$$

which reduced is $y^2=\frac{b^2}{a^2}(2ax-x^2)$, the equation to an ellipse, whose semi-axes are a and b.

365. *To determine the Nature of the curve which shall circumscribe any number of curves described according to a given law.*

Let $f(x, y, a)=0$, which may be put also in the form $y=\phi(x, a)$, be the equation to a curve whereof f or ϕ determines the species, and the parameter a the particular curve of that species, so that if a vary whilst f or ϕ remains the same, we shall pass from one curve to another of the same species: whence, if we differentiate the equation to the proposed family of curves, as it is called, with respect to a only, and then eliminate a, we shall obviously have an equation belonging to two *consecutive* curves, which not involving a, is therefore the general equation to the locus of the successive intersections of the curves; and the curve, which is the locus of all such intersections, touches all these curves, because any particular curve and the curve which is the locus of the consecutive intersections will manifestly have the same rectilineal

tangent. Hence, we have only to find the equation to a curve which touches any one of the given curves: to differentiate the equation so found with respect to its parameter, and then to eliminate that parameter by means of this equation and the one from which it is derived.

Ex. To find the curve which shall circumscribe any number of ellipses of given area, when their principal axes are all in the same directions.

Let a^2 be the given area, α and β the semi-axes of any individual ellipse; then, since $\pi\alpha\beta = a^2$, we have $\beta = \dfrac{a^2}{\pi\alpha}$, so that α is the variable parameter: and the equation

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1, \text{ becomes } \frac{x^2}{\alpha^2} + \frac{\pi^2\alpha^2y^2}{a^4} = 1:$$

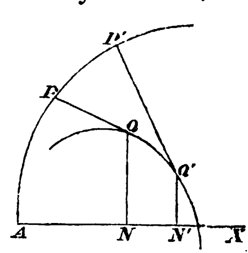
therefore, differentiating with respect to α, we get the equation

$$\frac{\pi^2y^2\alpha}{a^4} - \frac{x^2}{\alpha^3} = 0,$$

which belongs to this curve and its consecutive one: whence, eliminating α, we find $xy = \dfrac{a^2}{2\pi}$, for the equation to their intersection, whatever be the value of α: that is, the required curve is an hyperbola referred to its asymptotes.

366. *To find the length of the Evolute of any proposed part of a curve whose equation is given.*

Retaining the ordinary notation, we have seen in (186)



that the arc $QQ' = P'Q' - PQ$: whence it is obvious that the length of the evolute corresponding to any portion of a curve comprised between the points whose co-ordinates are given, will be equal to the difference of the radii of curvature at those points.

Ex. 1. To find the whole length of the evolute of an ellipse.

If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the equation from the centre, we have

$$\gamma = \frac{\{a^4 - (a^2 - b^2)x^2\}^{\frac{3}{2}}}{a^4 b};$$

$$\therefore \text{ when } x = 0, \text{ we have } P'Q' = \frac{a^2}{b}:$$

$$\text{and when } x = a, \text{ we find } PQ = \frac{b^2}{a}:$$

whence the length of the evolute corresponding to one of the elliptic quadrants

$$= \frac{a^2}{b} - \frac{b^2}{a} = \frac{a^3 - b^3}{ab}: \text{ and therefore its whole length} = \frac{4(a^3 - b^3)}{ab}.$$

Ex. 2. Find the length of that part of the evolute of a parabola which is situated within it.

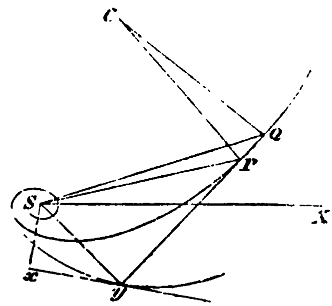
If $y^2 = 4ax$, we have $\gamma = \frac{2(x+a)^{\frac{3}{2}}}{\sqrt{a}}$: also the equation of the evolute is

$$y' = \pm \frac{2}{3\sqrt{3a}} (x' - 2a)^{\frac{3}{2}}:$$

and if $x' = x$ and $y' = y$, we find $x = 8a$: therefore the arc of each branch of the evolute $= P'Q' - PQ = 52a$.

367. *In a polar curve to find the Locus of the points of intersection of the tangent by the perpendicular let fall upon it from the pole.*

Let C be the intersection of the normals at the points



P and Q, $SP=r$, $Sy=p=r'$, $Sx=p'$ and θ' the angle described by Sy: then we shall manifestly have

$$\frac{d\theta'}{d\theta} = \text{limit of } \frac{PCQ}{d\theta} = \frac{ds}{\gamma d\theta} = \frac{1}{\sqrt{r^2-p^2}}\,\frac{dp}{d\theta},$$

by (190) and (205): whence is readily obtained

$$\frac{d\theta'}{dr'} = \frac{d\theta'}{d\theta}\,\frac{d\theta}{dr'} = \frac{d\theta'}{d\theta}\,\frac{d\theta}{dp} = \frac{1}{\sqrt{r^2-r'^2}};$$

$$\text{but } \frac{d\theta'}{dr'} = \frac{p'}{r'\sqrt{r'^2-p'^2}}, \text{ by (195);}$$

$$\therefore \frac{p'}{r'\sqrt{r'^2-p'^2}} = \frac{1}{\sqrt{r^2-r'^2}}, \text{ or } p' = \frac{r'^2}{r}:$$

and by the elimination of r, the equation of the locus proposed becomes known.

368. Cor. Since $p' = \dfrac{r'^2}{r}$, we have $r : r' :: r' : p'$; that is, $SP : Sy :: Sy : Sx$; and the angles SyP, Sxy being right angles, it appears that the triangles SPy, Syx are similar.

Ex. 1. In an Ellipse about the focus we have seen that

$$p = a\sqrt{1-e^2}\,\sqrt{\frac{r}{2a-r}}:$$

$$\text{from which } r = \frac{2ap^2}{a^2(1-e^2)+p^2} = \frac{2ar'^2}{a^2(1-e^2)+r'^2}:$$

therefore $p' = \dfrac{a^2(1-e^2)+r'^2}{2a}$, which is the equation of a circle whose radius is a, and the distance of whose centre from the focus of the ellipse is ae.

Ex. 2. In the Logarithmic Spiral $p = \dfrac{r}{\sqrt{1+k^2}}$: whence

$$\text{we have } r = p\sqrt{1+k^2} = r'\sqrt{1+k^2},$$

$$\text{and } p' = \frac{r'^2}{r'\sqrt{1+k^2}} = \frac{r'}{\sqrt{1+k^2}}:$$

or the locus of y is a Logarithmic Spiral similar and equal to the one proposed.

369. We may hence determine also the equations of the loci of similar intersections in the succeeding curves.

For, let r be the radius vector of the original curve, p the perpendicular upon its tangent; r_1, p_1; r_2, p_2, &c.......r_n, p_n, those of succeeding loci in order; then we have

$$p_1 = \frac{r_1^2}{r},\ p_2 = \frac{r_2^2}{r_1},\ p_3 = \frac{r_3^2}{r_2},\ \text{\&c.},\ p_n = \frac{r_n^2}{r_{n-1}}:$$

by means of which the equation of the n^{th} locus may be determined.

370. Cor. Hence it follows that the distances r, r_1, r_2, r_3, &c. are in a decreasing geometrical progression, and make equal angles with each other, so that their extremities will all be found in the arc of an equiangular spiral.

Ex. In a circle with the pole in its circumference, we have $p = \dfrac{r^2}{2a}$; wherefore $r = \sqrt{2ap} = \sqrt{2ar_1}$:

$$p_1 = \frac{r_1^2}{r} = \frac{r_1^{\frac{3}{2}}}{\sqrt{2a}}; \text{ whence } r_1^{\frac{3}{2}} = p_1\sqrt{2a} = r_2\sqrt{2a}:$$

$$p_2 = \frac{r_2^2}{r_1} = \frac{r_2^{\frac{4}{3}}}{\sqrt[3]{2a}}; \text{ whence } r_2^{\frac{4}{3}} = p_2\sqrt[3]{2a} = r_3\sqrt[3]{2a}:$$

$$p_3 = \frac{r_3^2}{r_2} = \frac{r_3^{\frac{5}{4}}}{\sqrt[4]{2a}}; \text{ whence } r_3^{\frac{5}{4}} = p_3\sqrt[4]{2a} = r_4\sqrt[4]{2a}:$$

and therefore generally, $p_n = \dfrac{r_n^2}{r_{n-1}} = \left(\dfrac{r_n^{n+2}}{2a}\right)^{\frac{1}{n+1}}$.

371. *To find the Locus of the points of intersection of the polar sub-tangent and tangent.*

From article (196) it appears that $ST = \dfrac{r^2 d\theta}{dr} = f(\theta)$; wherefore if we put $ST = r$ and $\angle TSX = \theta_,$, since it is manifest that

$$\angle TSX + \angle PSX = \frac{\pi}{2}, \text{ or } \theta_, + \theta = \frac{\pi}{2},$$

we shall have

$$r_, = f(\theta) = f\left(\frac{\pi}{2} - \theta_,\right) \text{ for the equation required.}$$

Also, after a similar manner may the equation of the locus of the intersection of the sub-tangent and tangent of this last spiral be found. Similarly of the intersection of the normal and sub-normal.

Ex. In the Reciprocal Spiral $r = a\theta^{-1}$; wherefore $ST = -a$, and consequently $r_, = -a$, which is the equation to a circle.

372. *To express the cosine of the multiple of an arc in descending powers of the cosine of the arc itself.*

Let $2\cos a = y + \frac{1}{y}$; then it is well known that

$$2\cos ma = y^m + \frac{1}{y^m}:$$

hence from the equation $2\cos a - y - \frac{1}{y} = 0$, we must find the values of y^m and y^{-m} by *Lagrange's* Theorem, and if $2\cos a = a$, the ordinary process gives

$$y^m = a^m - m a^{m-2} + \frac{m(m-3)}{1.2} a^{m-4} - \&c.,$$

$$y^{-m} = a^{-m} + m a^{-m-2} + \frac{m(m+3)}{1.2} a^{-m-4} + \&c.:$$

whence we have the value of $2\cos ma$ expressed by the sum of these two series, whatever be the value of m whether integral or fractional.

If m be a whole number, the value of $2\cos ma$ will manifestly be obtained from the first series by stopping at the term which involves a^0, since the remaining terms are the same as those of the second series, but with a different algebraical sign.

373. *To find the content of the greatest Cube that can be inscribed in a given ellipsoid.*

Let $2x$, $2y$, $2z$ be the sides of the required cube, which will therefore be, by the nature of the surface, subject to the condition expressed by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

if a, b, c be the semi-axes of the ellipsoid:

also the volume of the cube $u = 8xyz = a$ *maximum*:

whence taking the partial differentials we have from the former

$$\frac{x}{a^2} + \frac{z}{c^2}\frac{dz}{dx} = 0, \quad \frac{y}{b^2} + \frac{z}{c^2}\frac{dz}{dy} = 0;$$

and from the latter we obtain

$$\frac{du}{dx} = 8yz + 8xy\frac{dz}{dx} = 0, \text{ or } \frac{dz}{dx} = -\frac{z}{x},$$

$$\frac{du}{dy} = 8xz + 8xy\frac{dz}{dy} = 0, \text{ or } \frac{dz}{dy} = -\frac{z}{y};$$

whence by substitution we find $\frac{x^2}{a^2} = \frac{z^2}{c^2}$ and $\frac{y^2}{b^2} = \frac{z^2}{c^2}$,

and the equation of condition therefore gives

$$x = \frac{a}{\sqrt{3}},\ y = \frac{b}{\sqrt{3}} \text{ and } z = \frac{c}{\sqrt{3}}:$$

and the content of the greatest cube $= \frac{8abc}{3\sqrt{3}}$.

374. *To find when the Volume of the pyramid formed by the tangent plane to a curve surface, and the three co-ordinate planes is the least possible.*

Let $\frac{x}{a} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$, be the equation to any tangent plane; then are a, β, γ the distances from the origin at which the plane meets the co-ordinate axes: and the volume of the pyramid will be expressed by $\frac{1}{6}a\beta\gamma$:

whence we have $u = \frac{a^2\beta^2 z}{a\beta - \beta x - ay}$ *a minimum:*

also the characteristic equations $\frac{du}{da} = 0$ and $\frac{du}{d\beta} = 0$ give

$$2(a\beta - \beta x - ay) - a(\beta - y) = 0,$$

$$2(a\beta - \beta x - ay) - \beta(a - x) = 0,$$

from which it easily appears that

$$a = 3x,\ \beta = 3y, \text{ and therefore } \gamma = 3z.$$

If the equation to the curve surface be given, the actual value of the least volume may hence be determined.

375. *To investigate the Nature of the surface which shall envelope any number of surfaces of a given species described after a given law.*

Let $u = f(x, y, z, a) = 0$, be the equation to one of the proposed surfaces, the individual surface being defined by the particular magnitude of the parameter a: then, reasoning as in (364) and (365), we shall have, for the consecutive surface, $\frac{du}{da} = 0$: and if from this and the equation proposed the varying parameter a be eliminated, the resulting equation will manifestly be that of the required *Envelope*.

Ex. Required the surface which shall envelope all right cones of given volume having their axes in the same straight line, and their bases in the same plane.

Let $az = \beta(a - \sqrt{x^2 + y^2})$, be the equation to any one of the cones of the proposed magnitude: then, if the given volume $= a^3$, we shall have by (20), of the Introductory Chapter,

$$\tfrac{1}{3}\pi a^2 \beta = a^3, \text{ or } \beta = \frac{3a^3}{\pi a^2}:$$

$$\text{and this gives } z = \frac{3a^3}{\pi a^3}(a - \sqrt{x^2 + y^2}):$$

$$\therefore \frac{du}{da} = \frac{9a^3\sqrt{x^2 + y^2}}{\pi a^4} - \frac{6a^3}{\pi a^3} = 0, \text{ or } a = \frac{3}{2}\sqrt{x^2 + y^2}:$$

whence, by substitution, the equation above given becomes

$$z = \frac{4a^3}{9\pi(x^2 + y^2)};$$

which is therefore the equation to the required enveloping surface.

376. Cor. It follows from the same mode of reasoning, that if the surfaces of the proposed species involve more than one independent parameter α, β, &c., they must be eliminated by means of the equations

$$u=0, \quad \frac{du}{d\alpha}=0, \quad \frac{du}{d\beta}=0, \quad \&c.,$$

in order to obtain the equation to the surface which envelopes them all.

Ex. Let the surface of the given species be a plane defined by the equation

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1,$$

subject to the equation of condition $\alpha^n+\beta^n+\gamma^n=a^n$.

Here putting the equation in a different form, we have

$$u=\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}-1=0, \quad \frac{du}{d\alpha}=0 \text{ and } \frac{du}{d\beta}=0,$$

to eliminate the two independent parameters α, β:

$$\text{now } \frac{du}{d\alpha}=-\frac{x}{\alpha^2}-\frac{z}{\gamma^2}\frac{d\gamma}{d\alpha}=0,$$

$$\text{and } \frac{du}{d\beta}=-\frac{y}{\beta^2}-\frac{z}{\gamma^2}\frac{d\gamma}{d\beta}=0:$$

but, since $\gamma^n=a^n-\alpha^n-\beta^n$, we shall have

$$\frac{d\gamma}{d\alpha}=-\frac{\alpha^{n-1}}{\gamma^{n-1}} \text{ and } \frac{d\gamma}{d\beta}=-\frac{\beta^{n-1}}{\gamma^{n-1}};$$

whence, eliminating $\frac{d\gamma}{d\alpha}$ and $\frac{d\gamma}{d\beta}$, we find

$$\frac{x}{\alpha^2}=\frac{\alpha^{n-1}z}{\gamma^{n+1}} \text{ and } \frac{y}{\beta^2}=\frac{\beta^{n-1}z}{\gamma^{n+1}}:$$

and therefore $1 = \dfrac{x}{\alpha} + \dfrac{y}{\beta} + \dfrac{z}{\gamma} = \dfrac{(\alpha^n + \beta^n + \gamma^n)\,z}{\gamma^{n+1}} = \dfrac{a^n z}{\gamma^{n+1}}$,

$$\text{or } \gamma = a^{\frac{n}{n+1}} z^{\frac{1}{n+1}}:$$

from which we obtain $\alpha = a^{\frac{n}{n+1}} x^{\frac{1}{n+1}}$ and $\beta = a^{\frac{n}{n+1}} y^{\frac{1}{n+1}}$:

and by substituting these in the equation to the plane, we have

$$\left(\frac{x}{a}\right)^{\frac{n}{n+1}} + \left(\frac{y}{a}\right)^{\frac{n}{n+1}} + \left(\frac{z}{a}\right)^{\frac{n}{n+1}} = 1,$$

for the equation to the enveloping surface.

If -2 be the value of n, the surface touching all the planes will be that of a sphere, whose equation is

$$x^2 + y^2 + z^2 = a^2.$$

377. *To express the Radius of Curvature of any section of a curve surface in terms of its partial differential coefficients.*

Let the surface be referred to the tangent plane and two other planes perpendicular to it passing through the normal considered as the axis of z: then, if a normal plane be drawn so as to contain the proposed section, and v and z be the co-ordinates belonging to this plane, we shall have from (163)

$$\gamma = \frac{\left(1 + \dfrac{dz^2}{dv^2}\right)^{\frac{3}{2}}}{\dfrac{d^2z}{dv^2}}:$$

but in this case $\gamma = \dfrac{1}{\left(\dfrac{d^2z}{dv^2}\right)}$, since the plane of xy is a tangent plane, and therefore $\dfrac{dz}{dv} = 0$:

whence if a be the angle made by this normal plane with that of xz we shall obviously have

$$v = \frac{x}{\cos a} = \frac{y}{\sin a}:$$

$$\text{and therefore } \frac{dx}{dv} = \cos a \text{ and } \frac{dy}{dv} = \sin a:$$

but z being a function of x, y, it follows that z is also a function of v: hence from (268) we get

$$\frac{dz}{dv} = \frac{dz}{dx}\frac{dx}{dv} + \frac{dz}{dy}\frac{dy}{dv} = \frac{dz}{dx}\cos a + \frac{dz}{dy}\sin a:$$

$$\text{and thence we find } \frac{d^2z}{dv^2}$$

$$= \cos a\left(\frac{d^2z}{dx^2}\frac{dx}{dv} + \frac{d^2z}{dxdy}\frac{dy}{dv}\right) + \sin a\left(\frac{d^2z}{dy^2}\frac{dy}{dv} + \frac{d^2z}{dydx}\frac{dx}{dv}\right)$$

$$= \cos^2 a\frac{d^2z}{dx^2} + 2\sin a\cos a\frac{d^2z}{dxdy} + \sin^2 a\frac{d^2z}{dy^2} = \frac{1}{\gamma};$$

wherefore γ is expressed in the terms proposed.

378. *To find the Radius of absolute Curvature of a curve of double curvature, the arc of the curve being considered the principal variable.*

Let x', y', z' be the cosines of the angles made by a tangent to the curve at any point with the co-ordinate planes; $x'+h'$, $y'+k'$, $z'+l'$, those of the angles by a tangent at a point near to the former: then, if θ be the angle between the normals at these points, we shall have

$$\begin{aligned}\cos\theta &= x'(x'+h') + y'(y'+k') + z'(z'+l')\\ &= x'^2 + y'^2 + z'^2 + x'h' + y'k' + z'l'\\ &= 1 + x'h' + y'k' + z'l':\end{aligned}$$

$$\text{also}\quad (x'+h')^2+(y'+k')^2+(z'+l')^2=1\,;$$

$$\text{whence we find } 2\,(x'h'+y'k'+z'l')=-\,(h'^2+k'^2+l'^2):$$

$$\text{and}\quad \cos\theta=1-2\sin^2\frac{\theta}{2}=1-\frac{\theta^2}{2}\quad\text{ultimately}:$$

therefore from these two we have $\theta^2=h'^2+k'^2+l'^2$:

$$\text{hence the radius of curvature } \gamma=\frac{ds}{\theta}=\frac{ds}{\sqrt{h'^2+k'^2+l'^2}}:$$

$$\text{but by (340)}\quad x'=\frac{dx}{ds},\quad y'=\frac{dy}{ds}\quad\text{and}\quad z'=\frac{dz}{ds};$$

therefore, when the points of the proposed curve are consecutive,

$$h'^2=\left(\frac{d^2x}{ds}\right)^2,\quad k'^2=\left(\frac{d^2y}{ds}\right)^2,\quad l'^2=\left(\frac{d^2z}{ds}\right)^2:$$

whence by substitution we obtain

$$\gamma=\frac{ds}{\sqrt{\left(\frac{d^2x}{ds}\right)^2+\left(\frac{d^2y}{ds}\right)^2+\left(\frac{d^2z}{ds}\right)^2}}=$$

$$\frac{ds^2}{\sqrt{(d^2x)^2+(d^2y)^2+(d^2z)^2}}\quad\text{or}\quad\frac{1}{\sqrt{\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2z}{ds^2}\right)^2}},$$

which is analogous to the last formula of (182).

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